STABILITY OF THE SOLUTIONS OF FUZZY RELATION EQUATIONS

Part 2 : Undirectional Perturbation

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In this paper, the stability of the solutions of fuzzy relation equations is discussed by the method of the undirectional fuzzy pertuebation. First, the concepts of fuzzy perturbation matrix and fuzzy perturbation equation are advance, thus the stability of the solutions is defined. Next, the degree of stability of the solutions, a kind of metric of the stability, is given. An ordered quotient set is induced by use of the degree of stability, this eqotient set is of well chain characteristic, and equivalence classes, the element of this quotient set, are all of well partially ordered structure as well. Last, we present that the two open problems of the inverse problem of fuzzy multifactorial evaluation may be solved of the results obtained.

Keywords: Fuzzy perturbation matrix and equation, Stability of the solutions, Fuzzy relation equation.

1. The Stability of the Solutions

The stability of the solutions of fuzzy relation equation was first advance by the method of the directional fuzzy perturbation in paper (1), and to solve the two open problems of the inverse problem of fuzzy multifactorial evaluation was considered by use of the stability. In this paper, the stability of the solutions will be consider by the method of undirectional fuzzy perturbation.

Let $U=\{u_1,\dots,u_n\}$ and $V=\{v_1,\dots,v_m\}$ be two finite sets, and $F(U)$, $F(V)$ and $F(U\times V)$ be the family of all fuzzy sets on U, V and $U\times V$, respectively. We consider a fuzzy relation equation which regards $X\in F(U)$ as an unknown element when $B\in F(V)$, $R\in F(U\times V)$ are given:

$$X\circ R=B \tag{1.1}$$

i.e.

$$\bigvee_{1\le i\le n}(x_i\wedge r_{ij})=b_j \qquad j=1,\dots,m$$

Let $F(.)$ be the family of all fuzzy sets on any set. We define a partial ordering "$\leqslant$" in $F(.)$: $A_1\leqslant A_2$ iff $A_1\subset A_2$ for any $A_1, A_2\in F(.)$. In addition, it is denoted $A_1<A_2$ that $A_1\leqslant A_2$ and $A_1\neq A_2$.

Put $\mathcal{X}=\{X\in F(U)\mid X\circ R=B\}$, it is common knowledge that the partially ordered set $(\mathcal{X},\leqslant)$ is an infinite upper semilattice with the greatest element. We principally consider the change of the set of the solutions $\mathcal{X}$ by the perturbation.

Definition 1.1 Let $\varepsilon\in(0,1]$. $R^{\varepsilon}=(r_{ij}+\delta_{ij}\varphi(r_{ij})\varepsilon)_{n\times m}$ is called a ε-perturbation matrix of R. Where $\delta_{ij}=\pm1$; $\varphi=\eta\varphi_1+(1-\eta)\varphi_2$, $\eta\in[0,1]$, φ_1 and $\varphi_2\in[0,1]^{[0,1]}$ and satisfy conditions: φ_1 is strictly monotone increasing on $[0,1]$, φ_2 is strictly monotone increasing on $[0,1/2]$ and strictly monotone decreasing on $(1/2,1]$.

Remark. When $\delta_{ij}=-1$, it is possible that $r_{ij}-\varphi(r_{ij})\varepsilon<0$, i.e. $R^{\varepsilon}\notin F(U)$. But it do not hinder the discussion of us. Besides, the meaning of φ_1 and φ_2 is respectively: the larger the grade of membership, the larger the perturbation; the larger the grade of fuzzy, the larger the perturbation.

Definition 1.2 We call the equation:

$$Y\circ R^{\varepsilon}=B \tag{1.2}$$

which regards $Y\in F(U)$ as an unknown element ε-perturbation equation about the equation (1.1). The set of all the solutions of the equation (1.2) is denoted $\mathcal{Y}^{\varepsilon}$.

Definition 1.3 For any $X\in\mathcal{X}$, X is called ε-stable, about the equation (1.1), if it satisfies the equation (1.2), else, ε-unstable. The set of all ε-stable solutions is denoted $\mathcal{X}^{\varepsilon}$.

It is clear that $\mathcal{X}^{\varepsilon}=\mathcal{Y}^{\varepsilon}\cap\mathcal{X}$.

Definition 1.4 Let

$$a_{ij}=\begin{cases}\varphi(r_{ij}) & , \delta_{ij}=1\\ 0 & , \delta_{ij}=-1\end{cases} \qquad b_{ij}=\begin{cases}0 & , \delta_{ij}=1\\ \varphi(r_{ij}) & , \delta_{ij}=-1\end{cases}$$

we respectively call the matrixes

$$R^{\varepsilon^+}=(r_{ij}+a_{ij}\varepsilon)_{nxm} \quad , \quad R^{\varepsilon^-}=(r_{ij}-b_{ij}\varepsilon)_{nxm}$$

positive part matrix and negative part matrix of R, and respectively call the equations

$$YoR^{\varepsilon^+}=B \qquad YoR^{\varepsilon^-}=B$$

positive part equation and negative part equation of the equation (1.2). and their sets of all the solutions is respectively denoted $\mathcal{Y}^{\varepsilon^+}$ and $\mathcal{Y}^{\varepsilon^-}$.

Remark. $\varepsilon^{\pm}$-stability may respectively be defined like definition 1.3, and the sets of all $\varepsilon^{\pm}$-stable solutions is respectively denoted $\mathcal{X}^{\varepsilon^+}$ and $\mathcal{X}^{\varepsilon^-}$. Besides, it is clear that $\mathcal{X}^{\varepsilon^+}=\mathcal{Y}^{\varepsilon^+}\cap\mathcal{X}$ and $\mathcal{X}^{\varepsilon^-}=\mathcal{Y}^{\varepsilon^-}\cap\mathcal{X}$.

Proposition 1.1 $\mathcal{Y}^{\varepsilon^+}\cap\mathcal{Y}^{\varepsilon^-}=\mathcal{X}^{\varepsilon^+}\cap\mathcal{X}^{\varepsilon^-}$ ∎

Let $I_j^+=\{i \mid \delta_{ij}=1\}$ and $I_j^-=\{i \mid \delta_{ij}=-1\}$, we have

Proposition 1.2 (1) $I_j^+\cap I_j^-=\emptyset$ and $I_j^+ \cup I_j^-=\{1,\ldots,n\}$;

(2) $(\forall j)\ I_j^+=\emptyset \Rightarrow \mathcal{X}=\mathcal{X}^{\varepsilon^+}$ and $\mathcal{X}^{\varepsilon}=\mathcal{X}^{\varepsilon^-}$;

(3) $(\forall j)\ I_j^-=\emptyset \Rightarrow \mathcal{X}=\mathcal{X}^{\varepsilon^-}$ and $\mathcal{X}^{\varepsilon}=\mathcal{X}^{\varepsilon^+}$ ∎

Theorem 1.1 $\mathcal{X}^{\varepsilon^+}\cap\mathcal{X}^{\varepsilon^-}\subset\mathcal{X}^{\varepsilon}\subset\mathcal{X}^{\varepsilon^+}$ ∎

Proposition 1.3 (1) $\mathcal{X}^{\varepsilon^+}\cap\mathcal{X}^{\varepsilon^-}$ iff $\mathcal{X}^{\varepsilon^-}\supset\mathcal{X}^{\varepsilon}$;

(2) $(\forall j)(I_j^+=\emptyset$ or $I_j^-=\emptyset) \Longrightarrow \mathcal{X}^{\varepsilon^+}\cap\mathcal{X}^{\varepsilon^-}=\mathcal{X}^{\varepsilon}$ ∎

Proposition 1.4 $(\varepsilon,\eta\in(0,1], \varepsilon<\eta) \Longrightarrow \mathcal{X}^{\eta}\subset\mathcal{X}^{\varepsilon}$ (especially, $\mathcal{X}^{\eta^{\pm}}\subset\mathcal{X}^{\varepsilon^{\pm}}$) ∎

Proposition 1.5 If ε is appropriate small, then following four equalities are equivalent:

$$x_s\wedge r_{sj}=\bigvee_{1\le i\le n}(x_i\wedge r_{ij})$$

$$x_s \wedge (r_{sj}+a_{sj}\varepsilon) = \bigvee_{1\le i\le n} (x_i \wedge (r_{ij}+a_{ij}\varepsilon))$$

$$x_s \wedge (r_{sj}-b_{sj}\varepsilon) = \bigvee_{1\le i\le n} (x_i \wedge (r_{ij}-b_{ij}\varepsilon))$$

$$x_s \wedge (r_{sj}+\delta_{sj}\varphi(r_{sj})\varepsilon) = \bigvee_{1\le i\le n} (x_i \wedge (r_{ij}+\delta_{ij}\varphi(r_{ij})\varepsilon)) \quad \blacksquare$$

<u>Theorem 1.2</u> If ε is appropriate small, then $\mathcal{X}^{\varepsilon} \subset \mathcal{X}^{\varepsilon^-}$. ∎

2. Metric of the Stability

<u>Definition 2.1</u> Let $W(X) \triangleq \{\varepsilon\in(0,1] \mid X o R^{\varepsilon} = B\}$ for any $X\in\mathcal{X}$. Take mapping $S: \mathcal{X} \to [0,1]$ such that

$$S(X) = \begin{cases} \sup W(X) & , \quad W(X)\neq\emptyset \\ 0 & , \quad W(X)=\emptyset \end{cases}$$

$S(X)$ is called the degree of stability of X, about the equation (1.1).

<u>Definition 2.2</u> For any $X\in\mathcal{X}$, let

$$W^+(X) \triangleq \{\varepsilon\in(0,1] \mid X o R^{\varepsilon^+} = B\}$$

$$W^-(X) \triangleq \{\varepsilon\in(0,1] \mid X o R^{\varepsilon^-} = B\}$$

Take the mapping $S^+, S^-: \mathcal{X} \longrightarrow [0,1]$ such that

$$S^+(X) = \begin{cases} \sup W^+(X) & , \quad W^+(X)\neq\emptyset \\ 0 & , \quad W^+(X)=\emptyset \end{cases}$$

$$S^-(X) = \begin{cases} \sup W^-(X) & , \quad W^-(X)\neq\emptyset \\ 0 & , \quad W^-(X)=\emptyset \end{cases}$$

$S^+(X)$ and $S^-(X)$ is respectively called positive direction degree of stability and negative direction degree of stability, of X, about the equation (1.1).

<u>Proposition 2.1</u> S^- is an isotone mapping and S^+ is a inverse isotone mapping. ∎

<u>Proposition 2.2</u> (1) $S^+(X)\le S^-(X)$ iff $W^+(X)\subset W^-(X)$;

(2) $S^-(X)\le S^+(X)$ iff $W^-(X)\subset W^+(X)$;

(3) $S(X)\le S^+(X)$ iff $W(X)\subset W^+(X)$;

(4) $S(X)\le S^-(X)$ iff $W(X)\subset W^-(X)$. ∎

Theorem 2.1 $S^+(X)\wedge S^-(X)\leq S(X)\leq S^+(X)$ $(\forall X\in\mathcal{X})$ ∎

Corollary (1) $S^+(X)\leq S^-(X)$ iff $S(X)=S^+(X)$;
(2) $S^-(X)\leq S^+(X)$ iff $S(X)=S^-(X)$;
(3) $S^+(X)=S^-(X)$ iff $S^+(X)=S(X)=S^-(X)$. ∎

Proposition 2.3 If $X\leq Y$ for any $X,Y\in\mathcal{X}$, then
(1) $S(X)\leq S^+(X) \Rightarrow S(X)\leq S(Y)$;
(2) $S(Y)\leq S^-(X) \Rightarrow S(Y)\leq S(X)$. ∎

Proposition 2.4 If $X\leq Y\leq Z$ for any X,Y and $Z\in\mathcal{X}$, then
(1) $S(X)\leq S(Z) \Rightarrow S(X)\leq S(Y)$;
(2) $S(X)\geq S(Z) \Rightarrow S(Z)\leq S(Y)$;
(3) $S(X)=S(Z) \Rightarrow S(X)\vee S(Z)\leq S(Y)$. ∎

Definition 2.3 For any $X\in\mathcal{X}$ and $j\in\{1,\ldots,m\}$, put

$$W_j(X)\triangleq\{\varepsilon\in(0,1] \mid \bigvee_{1\leq i\leq n}(x_i\wedge(r_{ij}+\delta_{ij}\varphi(r_{ij})\varepsilon))=b_j\}$$

Take mapping $S_j:\mathcal{X}\to[0,1]$ such that

$$S_j(X)=\begin{cases}\sup W_j(X), & W_j(X)\neq\emptyset\\ 0, & W_j(X)=\emptyset\end{cases}$$

$S_j(X)$ is called degree of part stability of X, about equation (1.1). Positive direction degree of part stability $S_j^+(X)$ and negative direction degree of part stability $S_j^-(X)$ is respectively defined like the above.

Proposition 2.5 $S(X) = \bigwedge_{1\leq j\leq m} S_j(X)$ $(\forall X\in\mathcal{X})$ ∎

Definition 2.4 For any $X\in\mathcal{X}$ and any $j\in\{1,\ldots,m\}$, put

$$T_j\triangleq\{t \mid x_t\wedge r_{tj}=b_j\}$$

For any $t\in T_j$, put

$$W_{tj}(X)\triangleq\{\varepsilon\in(0,1] \mid x_t\wedge(r_{tj}+\delta_{tj}\varphi(r_{tj})\varepsilon)=b_j\}$$

Take mapping $S_{tj}:\mathcal{X}\to[0,1]$ such that

$$S_{tj}(X)=\begin{cases}\sup W_{tj}(X), & W_{tj}(X)\neq\emptyset\\ 0, & W_{tj}(X)=\emptyset\end{cases}$$

$S_{tj}(X)$ is called degree of subpart stability of X, about the equation (1.1). Positive direction degree of subpart stability $S^+_{tj}(X)$ and negative direction degree of subpart stability $S^-_{tj}(X)$ is respectively defined like the above.

<u>Definition 2.5</u> For any $X \in \mathcal{X}$ and $j \in \{1,\dots,m\}$, put

$$K_j \triangleq \{k \mid x_k \wedge r_{kj} < b_j\}$$

For any $k \in K_j$, put

$$H_{kj}(X) \triangleq \{\varepsilon \in (0,1] \mid x_k \wedge (r_{kj} + \delta_{kj}\varphi(r_{kj})\varepsilon) = b_j\}$$

Take mapping $E_{kj}: \mathcal{X} \longrightarrow [0,1]$ such that

$$E_{kj}(X) = \begin{cases} \sup H_{kj}(X) & , \quad H_{kj}(X) \neq \emptyset \\ 0 & , \quad H_{kj}(X) = \emptyset \end{cases}$$

$E_{kj}(X)$ is called degree of subpart extension of X, about the equation (1.1). Besides, take mapping $E_j: \mathcal{X} \longrightarrow [0,1]$ such that

$$E_j(X) = \bigvee_{k \in K_j} E_{kj}(X)$$

$E_j(X)$ is called degree of part extension of X, about the equation (1.1).

<u>Proposition 2.6</u> For any $X \in \mathcal{X}$, we have

(1) $$S_{tj}(X) = \begin{cases} S^+_{tj}(X) & , \quad \delta_{tj} = 1 \\ S^-_{tj}(X) & , \quad \delta_{tj} = -1 \end{cases}$$

(2) $$S_j(X) = (\bigvee_{t \in T_j} S_{tj}(X)) \vee E_j(X)$$

(3) $$S(X) = \bigwedge_{1 \le j \le m} ((\bigvee_{t \in T_j} S_{tj}(X)) \vee (\bigvee_{k \in K_j} E_{kj}(X)))$$ ∎

3. Three Ordered Quotient Sets

Three equivalence relations E, E^+ and E^- may respectively be determinied by the mapping S, S^+ and S^-: for any $X, Y \in \mathcal{X}$

$$XEY \quad \text{iff} \quad S(X) = S(Y)$$

$$XE^{+}Y \quad \text{iff} \quad S^{+}(X)=S^{-}(Y)$$

$$XE^{-}Y \quad \text{iff} \quad S^{+}(X)=S^{-}(Y)$$

Thus we obtain three quotient sets of $\mathcal{X}$:

$$\mathcal{X}/E=\{\bar{X} \mid X\in\mathcal{X}\},\ \mathcal{X}/E^{+}=\{\check{X} \mid X\in\mathcal{X}\},\ \mathcal{X}/E^{-}=\{\hat{X} \mid X\in\mathcal{X}\}$$

where $\bar{X},\check{X}$ and $\hat{X}$ are all equivalence classes for X.

$\mathcal{X}/E$, $\mathcal{X}/E^{+}$ and $\mathcal{X}/E^{-}$ is respectively ordered by $\dashv$, $\dashv$ and $\prec$:
For any $X,Y\in\mathcal{X}$

$$\bar{X}\dashv\bar{Y} \quad \text{iff} \quad S(X)\leq S(Y)$$

$$X \dashv Y \quad \text{iff} \quad S^{+}(X)\leq S^{+}(Y)$$

$$X \prec Y \quad \text{iff} \quad S^{-}(X)\leq S^{-}(Y)$$

Thus we obtain three ordered quotient sets:

$$(\mathcal{X}/E,\dashv),\quad (\mathcal{X}/E^{+},\dashv),\quad (\mathcal{X}/E^{-},\prec)$$

Theorem 3.1 $(\mathcal{X}/E,\dashv)$ is a finite chain.

Proof. We only need to prove the image set of S on $\mathcal{X}$, $S(\mathcal{X})$, is a finite set. It is can be seen that

$$S(\mathcal{X})\subset\bigcup_{j=1}^{m}S_j(\mathcal{X})\subset\bigcup_{j=1}^{m}\Big(\Big(\bigcup_{t\in T_j}S_{tj}(\mathcal{X})\Big)\cup\Big(\bigcup_{k\in K_j}E_{kj}(\mathcal{X})\Big)\Big)$$

Thus we only need to prove that $S_{tj}(\mathcal{X})$ and $E_{kj}(\mathcal{X})$ are all finite sets. It is easy to obtain

$$S_{tj}(\mathcal{X})\subset\begin{cases}\{0,1,(r_{tj}-b_j)/\varphi(r_{tj})\}, & \varphi(r_{tj})\neq 0\\ \{1\}, & \varphi(r_{tj})=0\end{cases}$$

$$E_{kj}(\mathcal{X})\subset\begin{cases}\{0,1,(b_j-r_{kj})/\varphi(r_{kj})\}, & \varphi(r_{kj})\neq 0\\ \{0\}, & \varphi(r_{kj})=0\end{cases}$$

Hence $S(\mathcal{X})$ is a finite set. ▮

Corollary Put

$$A_j=\{(r_{tj}-b_j)/\varphi(r_{tj}) \mid t\in T_j,\ \varphi(r_{tj})\neq 0\}$$

$$B_j=\{(b_j-r_{tj})/\varphi(r_{kj}) \mid k\in K_j,\ \varphi(r_{kj})=0\}$$

then $J(\mathcal{X}) \subset (\bigcup_{j=1}^{m}(A_j \cup B_j)) \cup \{0,1\}$ ∎

Theorem 3.2 (1) $(\mathcal{X}/E^+, \dashv)$ is a finite chain, and its the greatest element contains some minimal elements of $\mathcal{X}$ and its least element contains the greatest element of $\mathcal{X}$;

(2) $(\mathcal{X}/E^-, \prec)$ is a finite chain, and its the greatest element contains the greatest element of $\mathcal{X}$ and its the least element contains some minimal elements of $\mathcal{X}$. ∎

4. Partially Ordered Structure of $\overline{X}$, $\tilde{X}$ and $\hat{X}$

A nonempty partially order set $(P, \leqslant)$ is called upper (lower) inductiue if every chain of P has a upper (lower) bound.

Definition 4.1 Let $(P, \leqslant)$ be a nonempty partially ordered set and P^* (P_*) be the of all maximal (minimal) elements of P. P is possessed of maximal (minimal) character if $\forall a \in P\ \exists b \in P^*(P_*)$ such that $a \leqslant (\geqslant) b$.

Proposition 4.1 That P is upper (lower) inductiue implies that P is possessed of maximal (minimal) character. ∎

Proposition 4.2 For any chain $\mathcal{A}$, of $\mathcal{X}$, the least upper bound $\vee\mathcal{A}$ and the greatest lower bound $\wedge\mathcal{A}$ of $\mathcal{A}$ belong to $\mathcal{A}$. ∎

Proposition 4.3 $\mathcal{X}$ is both upper inductiue and lower inductiue, hence $\mathcal{X}$ is of both maximal character and minimal character. ∎

Theorem 4.1 For any $\overline{X} \in \mathcal{X}/E$, $\overline{X}$ is both upper inductiue and lower inductiue, hence $\overline{X}$ is of both maximal character and minimal character. ∎

Corollary For any $\tilde{X} \in \mathcal{X}/E^+$ and $\hat{X} \in \mathcal{X}/E^-$, $\tilde{X}$ and $\hat{X}$ are all both upper inductiue and lower inductiue. Hence they are all both character with maximum and character minimum. ∎

It is very important that $\overline{X}$ is of maximal character and minimal character. They have important applications in the inverse problem of fuzzy multifactorial evaluation.

5. Inverse Problem of Multifactorial Evaluation

It is common knowledge that inverse problem of fuzzy multifactorial evaluation is expressed as the equation (1.1). But there are two problems which need to study thoroughly.

Problem 1 : How to choose an element X_o from $\mathcal{X}$ as the distribution of the weight numbers of the evaluation process when $|\mathcal{X}|>1$?

Problem 2 : How to choose $A_o \in F(U)$ as the distribution of the weight numbers when $\mathcal{X}=\emptyset$?

The two problems was discussed in detail by use of the stability of the solutions of fuzzy relation equations in the paper (1). In fact, the stability of the solutions of the paper (1) is the cases when $\delta_{ij}\equiv 1$ and $\delta_{ij}\equiv -1$. (see the paper (1))

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