THE DEPENDENCE OF A SYSTEM OF FUZZY VECTORS AND THE RANK OF A FUZZY MATRIX

Zha Jian-lu
(Dalian Fisheries College, China)

Abstract

In this paper, the author has discussed the dependence of a system of fuzzy vectors and the rank of a fuzzy matrix over a some-ring $I=([0,1], a+b=\max\{a,b\}, ab=\min\{a,b\})$, and achieved the following essential results:

1. The author has further proved the uniqueness of the cardinal number of the maximum independent group in a system S of a fuzzy vector up to any spanning set of its generated sub-space $\langle S\rangle$ and it is also the minimum spanning set of $\langle S\rangle$.

2. The paper has given the general direction and steps in determing the row (column) rank of a given fuzzy matrix.

1. Fundamental Concepts.

In this paper we discuss fuzzy vector and fuzzy matrix over commutative semi-ring $I=([0,1], a+b=\max\{a,b\}, ab=\min\{a,b\})$.

Let $A=[a_{ij}]_{n\times m}$, $B=[b_{ij}]_{n\times m}$, $C=[c_{jk}]_{m\times l}$.

From D. Dubois and H. Prade[2], we quote the following definitions:

Equality of matrices $A=B$ iff $(\forall i)(\forall j)\ a_{ij}=b_{ij}$.

Addition of matrices $A+B=[a_{ij}+b_{ij}]_{n\times m}$.

Multiplication of a matrix by a number $\lambda A=[\lambda a_{ij}]_{n\times m}$ where $\lambda\in[0,1]$

Multiplication of matrices $AC=[d_{ik}]_{n\times l}$, where $d_{ik}=\sum_{j=1}^{m} a_{ij}c_{jk}$.

Multiplication of matrices by operator ⓐ $A ⓐ C=[e_{ik}]_{n\times l}$

where $e_{ik}=\prod_{j=1}^{m}(a_{ij}\alpha c_{jk})$, $a_{ij}\alpha c_{jk}=\begin{cases}1 & a_{ij}\le c_{jk}\\ c_{jk} & a_{ij}>c_{jk}\end{cases}$.

Transpose of a matrix. $A^T=[a_{ji}]_{m\times n}$.

Computation shows that the operations mentioned above may satisfy the properties of the corresponding in ordinary matrix.

Taking fuzzy vectors to be special case of fuzzy matrix, its operations will not be defined again.

It is obvious that the operation of addition of vectors and multiplication of vectors by a number is a closed system over the set V_n of all vectors of n-dim.

Definition 1-1. Let vectors system $S=\{x_1, -----, x_m\}$ and $X\in V_n$ if $\exists\lambda_i\in[0,1]$ such that $X=\sum_{i=1}^{m}\lambda_i X_i$, X is called a linear combination of S, or X is spanned from S. When every vector in the system A of fuzzy vector can be spanned from S, then A can be spanned from S. When A and S can spanned from each other, they are called an equivalence, and denoted as $A\sim S$.

Apparently, these relations have the nature of reflexive, symmetrical and transitive characters.

Definition 1-2. A subset W of V_n is called a subspace of V_n, if and only if W is a closed system for addition of vectors and multiplication of vector by a number.

The set of all linear combinations of S is denoted $\langle S\rangle=\{X \mid X=\sum\lambda_i X_i,\ X_i\in S,\ \lambda_i\in[0,1]\}$, and it is obviously a subspace of V_n.

If $\langle S\rangle=W$, then S is called a spanning set for W and subspace W is spanned from S.

Proposition 1-1. Let $A=\{a_1,\cdots,a_k\}$, $B=\{b_1,\cdots b_m\}$, then $A\sim B$ iff $\langle A\rangle=\langle B\rangle$.

Proof. Suppose $A\sim B$, then $\forall a\in\langle A\rangle\ \exists\alpha_i\in[0,1]$ such that $a=\sum_{i=1}^{k}\alpha_i a_i$,

$\forall a_i\in A\quad \exists\beta_{ij}\in[0,1]$ such that $a_i=\sum_{j=1}^{m}\beta_{ij}b_j$

There-by, $a=\sum_{i=1}^{k}(\alpha_i\sum_{j=1}^{m}\beta_{ij}b_j)=\sum_{j=1}^{m}(\sum_{i=1}^{k}\alpha_i\beta_{ij})b_j\in\langle B\rangle$, $\langle A\rangle\subseteq\langle B\rangle$.

Similarly, $B\subseteq A$, Hence $\langle A\rangle=\langle B\rangle$.

The converse is evident. ∎

Corollary 1. Two spanning sets of a subspace are equivalent to each other.

Corollary 2. Let $X\in V_n$, if X cannot be spanned by S, then X cannot be spanned by any spanning sets of $\langle S\rangle$.

II. The depandence of systems of fuzzy vectors.

Definition 2-1. Let $S\subset V_n$. S is independent if and only if $\forall X\in S$ is not a linear combination of $S\setminus\{X\}$. If $\exists X\in S$, X is a linear combination of $S\setminus\{X\}$, S is said to dependent while X is called a dependent element in S. Otherwise X is called an independent element in S.

Straightforward. If X is a dependent element in S, then $S\sim S\setminus\{X\}$.

By this Definition, we have,

Proposition 2-1. Let $S\subset V_n$. If a subset of S is dependent, then S is dependent.

Proposition 2-2. Let $S=\{X_1,\cdots,X_m\}\subset V_{n+1}$, $X_i=(x_{i1},\cdots,x_{in},x_{i(n+1)})$ and $S'=\{X'_1,\cdots,X'_m\}\subset V_n$, $X'_i=(x_{i1},\cdots,x_{in})$ $i=1,\cdots m$. If S is dependent, then S' is dependent. When $x_{i(n+1)}\le\min_{i,j}\{x_{ij}\}$ $j=1,\cdots n$ and S' is dependent, then

S is dependent.

Definition 2-2. Let $T \subseteq S \subset V_m$. If T is independent and X is a linear combination of T for $\forall X \in S$. T is called a maximum independent group of S.

By this definition, we have the following properties:

(1) A system of vectors and its maximum independent group are equivalent.

(2) Two maximum independent groups of S are equivalent.

(3) If $A \subseteq S$ and $A \sim S$, then a maximum independent group of A is the same of S.

In special case. If X is dependent element in S, then a maximum independent group of $S \setminus \{X\}$ is the same of S.

(4) If X is an independent element of S, then X is an element of all maximum independent groups of S.

Definition 2-3. Let $S=\{X_1, \cdots, X_m\} \subset V_n$, $X_K \in S$, X_K is a standard element if and only if when $X_K = \sum_{i=1}^{m} a_{Ki} X_i$, then $X_K = a_{KK} X_K$.

Proposition 2-3. Let $S=\{X_1, \cdots, X_m\} \subset V_n$, $X_K \in S$, and X_K is a standard element. If $X_K = \sum_{j=1}^{p} S_j$ where $S_j \in \langle S \rangle$, then $\exists S_l$ $(1 \le l \le p)$ such that $X_K = S_l$.

Proof. Since $\forall S_j \in \langle S \rangle$ $\exists a_{ji} \in [0,1]$ such that $S_j = \sum_{i=1}^{m} a_{ji} X_i$. Therefore,

$$X_K = \sum_{j=1}^{p} S_j = \sum_{j=1}^{p} \left(\sum_{i=1}^{m} a_{ji} X_i \right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{p} a_{ji} \right) X_i$$

. Then, by Definition 2-3, $X_K = \left(\sum_{j=1}^{p} a_{jK} \right) X_K$. Since $j = 1, \cdots, p$ (only finite values), therefore, $\exists a_{lK}$ $(1 \le l \le p)$, such that $X_K = a_{lK} X_K \le S_l$, but since $X_K = \sum_{j=1}^{p} S_j$, we have $X_K \ge S_l$. Hence $X_K = S_l$.

Proposition 2-4. Let $S=\{X_1, \cdots, X_m\} \subset V_n$, $X_j = (x_{1j}, \cdots, x_{nj})$. If S is independent, then there exist an independent system of vectors conposed by standard elements. This system of vectors in subspace$\langle S \rangle$ is equivalent to S and their cardinal numbers are also equivalent to each other.

Proof. If $\forall X_j \in S$; X_j is standard element, then this proposition is obviously terable.

If there is a non-standard element X_K in S, then $\exists a_j \in [0,1]$ such that $X_K = \sum_{j=1}^{m} a_j X_j$; and $X_K \ne a_K X_K$.

By the independence of S, we get $\{x_{iK} \mid x_{iK} = a_K x_{iK} \ge \sum_{j \ne K} a_j x_{ij}\} \ne \phi$. Suppose $x_{pK} = \max\{x_{iK} \mid x_{iK} = a_K x_{iK} \ge \sum_{j \ne K} a_j x_{ij}\}$, then $x_{pK} \in \{x_{iK}\}$, $x_{pK} \le a_K < \max\{x_{iK}\}$ and $X_K = \sum_{j \ne K} a_j X_j + x_{pK} X_K$.

Let $X_K^{(1)} = x_{pK} X_K$, $X_K^{(1)} = (x_{1K}^{(1)}, \cdots, x_{nK}^{(1)})$, then $x_{iK}^{(1)} \in \{x_{iK}\}$, and $x_{pK} = \max\{x_{iK}^{(1)}\} < \max\{x_{iK}\}$. Replacing X_K by $X_K^{(1)}$, then $S_1 = \{X_1^{(1)}, \cdots, X_m^{(1)}\}$

where $X_j^{(1)} = \begin{cases} X_j & j \neq k \\ x_{pk} X_k & j = k \end{cases}$.

It is easy to proof the following properties hold on S , i.e. $S_1 \sim S$, S_1 is independent and if X_i is a standard element for $X_i \in S$ $(i \neq k)$ then $X_i^{(1)}$ is still a standard element for $X_i^{(1)} \in S_1$, $|S_1| = |S|$.

If $X_k^{(1)}$ is a standard element, the replacement of X_k ceases.

If $X_k^{(1)}$ is still not a standard element, we may repeat the above replacement, we'll get $S_2 = \{X_1^{(2)}, \cdots, X_m^{(2)}\}$, $X_j^{(2)} = \begin{cases} X_j^{(1)} & j \neq k \\ x_{pk}^{(1)} X_k & j = k \end{cases}$, then $x_{pk}^{(1)} \in \{x_{ik}\}$, $x_{pk}^{(1)} = \max_i \{x_{ik}^{(2)}\} < \max_i \{x_{ik}^{(1)}\} = x_{pk}$ and the above properties of S_1 still hold for S_2, ---- etc. So long as no standard element appears in the previous replacements, then for every operation of replacement, there is $x_{pk} > x_{pk}^{(1)} > x_{pk}^{(2)} >$ ----- and $x_{pk}^{(t)} \in \{x_{ik}\}$. Therefore, this process of replacement for X_k must terminate after a finite number of operations, but this happens only when we have obtained a standard element $X_k^{(t)} \in S_t$ $(t \leq n-1)$.

Since S only contains finite number of non-standard elements, therefore, after we go through a finite number of the above steps of replacement, we are sure to obtain an independent system of vectors in $\langle S \rangle$, composed of standard elements equivalent to S and have the same cardinal numbers. ▮

Proposition 2-5. Let $S = \{X_1, \cdots, X_m\} \subset V_n$, $R = \{Y_1, \cdots, Y_l\} \subset V_n$. If S is independent and $S \sim R$, the $m \leq l$.

Proof. By Proposition 2-4, there exist an independent system of vectors S' consisting of standard elements in S , such that $S' \sim S$, $|S'| = |S|$.

Since $S' \sim S \sim R$, therefore, $\exists a_{ij} \in [0,1]$ such that $X_i' = \sum_{j=1}^{l} a_{ij} Y_j$ for $\forall X_i' \in S'$ since $a_{ij} Y_j \in \langle R \rangle = \langle S' \rangle$, by Proposition 2-3, $\forall X_i' \in S'$ $\exists j = k$ such that $X_i' = a_{ik} Y_k$.

If $m > l$, then there exist at least two vectors in S' that correspond to the same vector in R. It is contradictory to the independence of S', therefore $m \leq l$. ▮

Corollary 1. Any maximum independent group of a given system of vectors has a unique cardinality.

Corollary 2. The maximum independent group of any spanning set for finitely generated subspace has a unique cardinality.

Corollary 3. The maximum independent group of any spanning set for finitely generated subspace is the minimum spanning set for the subspace.

Definition 2-4. The minimum spanning set of a finitely generated subspace is called the basis of this subspace and its cardinal number is called the rank of this subspace.

III. The rank of a fuzzy matrix.

Definition 3-1. The subspace spanned from the row vectors group of a fuzzy matrix A is called row space R(A) of A, while the rank of R(A) is called the row rank of A, and is denoted by $\rho_r(A)$. Column space C(A) and column rank $\rho_c(A)$ are defined in similar way. If $\rho_r(A)=\rho_c(A)$, both are called the rank of A and is denoted by $\rho(A)$.

By Corollary of Proposition 2-5 and Definition 2-4, $\rho_r(A)$ and $\rho_c(A)$ are respectively the cardinality of maximum independent group for systems of row vectors of A, and systems of column vectors of A.

By 3-1, we can easily proof that the following propositions hold true.

Proposition 3-1. Let $A=[a_{ij}]_{n\times m}$, row vector r_k is a linear combination of $R\setminus\{r_k\} \Leftrightarrow (A_r^{(k)} \circledcirc r_k^T)^T A_r^{(k)} = r_k$ for $r_k \in R$.

Column vector c_i is a linear combination of $C\setminus\{c_i\} \Leftrightarrow A_c^{(i)}((A_c^{(i)})^T \circledcirc c_i) = c_i$ for $c_i \in C$ where $R=\{r_1,\cdots,r_n\}$, $r_i=(a_{i1},\cdots,a_{im})$, $A_r^{(k)}=\begin{pmatrix} r_1 \\ \vdots \\ r_{k-1} \\ r_{k+1} \\ \vdots \\ r_n \end{pmatrix}$

$C=\{c_1,\cdots,c_m\}$, $c_j=\begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix}$, $A_c^{(i)}=(c_1,\cdots,c_{i-1},c_{i+1},\cdots,c_m)$.

Corollary 1. If one of the components of row vector r_k is the unique maximum component of the corresponding column where the component of row vector lies, then r_k is an independent element in R.

Corollary 2. If the minimum component of a non-iso-element's row vector r_k is the unique minimum component of the corresponding column where the minimum component lies, the r_k is an independent element in R.

Corollary 3. Let $X=(x_1,\cdots,x_n)$; $Y=(y_1,\cdots,y_n)$, $\exists\lambda\in[0,1]$ such that $Y=\lambda X$ $\Longleftrightarrow$ $y_i \le x_i$ and $y_i=\max\{y_1,\cdots,y_n\}$ for all y_i of $y_i<x_i$ for $i=1,\cdots,n$.

We can easily obtain the dual statement for the above corollaries for column.

Definition 3-2. If all elements in a row (or column) of a matrix A are same, it is called an iso-elemental row (column).

When minimal elements of a matrix A sonstitute an iso-elemental row (column), this iso-elemental row (column) is considered to be cancellable and the matrix which contains the remaining rows(column) is called a submatrix of A.

Proposition 3-2. If an operation of consecutive cancellation over a given matrix A and its submatrix is exercised, we will obtain a $k\times l$ submatrix A', then $\rho_r(A)=\rho_r(A')$, $\rho_c(A)=\rho_c(A')$.

Proof. Suppose we consecutively cancel the matrix A and its sub-

[illegible] the resultant submatrix A' will locate in the upper left corner [illegible] A.

By Definition 3-2, it's easy to conceive:

(1) [illegible] elements in A' will not be less than all the cancelled elements;

(2) The cancelled elements: $a_{ij} = \begin{cases} a_{ij} & i \le k,\ j > l \\ a_{i1} a_{1j} & i > k,\ j > l \\ a_{i1} & i > k\ \ j \le l \end{cases}$

[illegible], when $j > l$, $a_{ij} \le a_{i1}$ for every element in C_j and for every [illegible] $a_{ij} = a_{1j} = \max\{a_{1j}, \cdots, a_{nj}\}$. By Corollary 3 of Proposition [illegible], we have $C_j = a_{1j} C_1$, so that the maximum independent group of $\{C_1, \cdots, C_l\}$ [illegible] the column vectors group C of A.

[illegible] $i > k$, $a_{ij} = a_{i1}$ and not exceeding the minimum element of A [illegible] Proposition 2-2, we have $\{C_1, \cdots, C_l\}$ and the column vectors [illegible] simultaneously dependent or independent. Therefore, the maximal independent group of the column vectors system of A' corresponds [illegible] the maximal independent group of $\{C_1, \cdots, C_l\}$. Thus, $\rho_c(A) = \rho_c(A')$

Similarly, we can proof $\rho_r(A) = \rho_r(A')$. ∎

From the above propositions and the properties of the maximal independent group, we obtain the steps for determing the row rank $\rho_r(A)$ of a fuzzy matrix A ;

(1) Cancel out if possible any given matrix A and obtain submatrix $A'_{k \times l}$. Let the row vectors group of the submatrix be $R = \{r_1, \cdots, r_k\}$.

(2) Applying the principles of Proposition 3-1 and its corollaries, check every $r_i \in R$ if it is a linear combination of $R \setminus \{r_i\}$

If every $r_i \in R$ are not linear combination of $R \setminus \{r_i\}$, the R itself is a maximal independent group.

If $r_s \in R$ is a linear combination of $R \setminus \{r_s\}$, repeat the above operation for $R \setminus \{r_s\}$. Evidently, we can obtain, after a finite number of such operation, a partial group of R, denoted as R', in which every row vector is the independent element of R', i.e. R' is a maximal independent group of R and $\rho_r(A) = |R'|$.

The operations for determing $\rho_c(A)$ are similar to the above.

For example,

$$A = \begin{pmatrix} 0.4 & 0.9 & 0.1 & 0.9 & 0.6 \\ 0.7 & 0.3 & 0.1 & 0.7 & 0.7 \\ 0.2 & 0.2 & 0.1 & 0.2 & 0.2 \\ 0.2 & 0.8 & 0.1 & 0.8 & 0.6 \\ 0.8 & 0.5 & 0.1 & 0.8 & 0.8 \end{pmatrix}$$

Determine $\rho_r(A)$ and $\rho_c(A)$.

Solution. Cancel consecutively the third row and column of A, we get

$$A' = \begin{pmatrix} 0.4 & 0.9 & 0.9 & 0.6 \\ 0.7 & 0.5 & 0.7 & 0.7 \\ 0.2 & 0.3 & 0.8 & 0.6 \\ 0.8 & 0.5 & 0.8 & 0.8 \end{pmatrix}$$

Since 0.9 of r_1 and 0.8 of r_4 are unique maximal components of C_2 and C_4 respectively, 0.3 of r_2 and 0.2 of r_3 are the unique minimal components of C_2 and C_1, respectively, therefore, $\forall r_i$ $(i=1,2,3,4)$ is not a linear combination of $R\setminus\{r_i\}$, so that $\rho_r(A)=4$.

Since the minimum components 0.2 of C_1 and 0.3 of C_2 are the unique minimal components of r_3 and r_2 respectively, therefore, C_1 and C_2 are not linear combinations of $C\setminus\{C_1\}$ and $C\setminus\{C_2\}$ respectively.

But $$\begin{pmatrix} 0.4 & 0.9 & 0.6 \\ 0.7 & 0.3 & 0.7 \\ 0.2 & 0.8 & 0.6 \\ 0.8 & 0.5 & 0.8 \end{pmatrix} \left[\begin{pmatrix} 0.4 & 0.7 & 0.2 & 0.8 \\ 0.9 & 0.3 & 0.8 & 0.5 \\ 0.6 & 0.7 & 0.6 & 0.8 \end{pmatrix} \circledast \begin{pmatrix} 0.9 \\ 0.7 \\ 0.8 \\ 0.8 \end{pmatrix} \right] = \begin{pmatrix} 0.9 \\ 0.7 \\ 0.8 \\ 0.8 \end{pmatrix}$$

therefore C_3 is a linear combination of $C\setminus\{C_3\}$.

$$\begin{pmatrix} 0.4 & 0.9 \\ 0.7 & 0.3 \\ 0.2 & 0.8 \\ 0.8 & 0.5 \end{pmatrix} \left[\begin{pmatrix} 0.4 & 0.7 & 0.2 & 0.8 \\ 0.9 & 0.3 & 0.8 & 0.5 \end{pmatrix} \circledast \begin{pmatrix} 0.6 \\ 0.7 \\ 0.6 \\ 0.8 \end{pmatrix} \right] = \begin{pmatrix} 0.6 \\ 0.7 \\ 0.6 \\ 0.8 \end{pmatrix}$$

therefore C_4 is a linear combination of $C\setminus\{C_3, C_4\}$ so that $\{C_1, C_2\}$ is the maximum independent group of A'. $\rho_c(A) = 2$.

Proposition 3-3. Let $A = [a_{ij}]_{n\times m}$, $\max_{i,j}\{a_{ij}\} = a_{kl}$, the following items are equivalent to each other.

(1) Consecutively cancel out matrix A and its submatrix, we get finally a submatrix A' with only one single row or column.

(2) $\rho_r(A) = \rho_c(A) = 1$

(3) $A = C_l r_k$

Proof. (1) $\Rightarrow$ (2). We may suppose that A' is a submatrix containing only one single column, we know by Proposition 3-2 that $\rho_c(A) = 1$. By definition 3-2 and $a_{kl} = \max_{i,j}\{a_{ij}\}$ we get $(\forall i)(\forall j)$ $a_{ij} = a_{il}a_{kj} \le a_{kj}$; and the [illegible] components $a_{ij} = a_{il} = \max\{a_{i1}, \cdots, a_{im}\}$ in which $a_{ij} < a_{kj}$; we will have by Proposition 3-1, Corollary 3, $(\forall i)$ $r_i = a_{il} r_k$ thus $\rho_r(A) = 1$.

([illegible]) $\Rightarrow$ (3) From $\rho_r(A) = \rho_c(A) = 1$, $a_{kl} = \max_{i,j}\{a_{ij}\}$, we know [illegible] a_{il} [illegible], so that $A = c_l r_k$.

([illegible]) $\Rightarrow$ ([illegible]) By $A = c_l r_k$, we get $a_{ij} = a_{il} a_{kj}$. Arrange these components a_{il} [illegible], a_{kj} ($j = 1, \cdots, m$) in order in accordance to their values ([illegible] arranged optionally if their values are equal), with the minimum [illegible] the left corner. Thus, when the minimum is one in the a_{il}, the corresponding row can be cancelled, and when it is in the a_{kj}, the corresponding column can be cancelled. Repeat the operation consecutively from left to right, we will have at last, after a finite number of such operations, a submatrix with only one single row or column. ∎

References

(1) K.H. Kim, F.W. Roush, "Generalized Fuzzy Matrices" ; "Fuzzy Sets and Systems". Vol. 4 (1980) No. 3, p.293-316.

(2) D. Dubois and H. Prade. "Fuzzy sets and Systems" 1980, Academic Press.