ON ANY CLASS OF FUZZY PREFERENCE RELATIONS IN REAL LINE

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At the first, the fuzzy preference relation is defined by means of weak notions [2] . Furthermore, the fuzzy relation "less or equal" is presented as fuzzy preference relation in real line. Its definition follows from well-known properties of analogous crisp relation. The connections between the fuzzy relation "less or equal" and the fuzzy relation "less than" generated by first are showed.

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1. Introduction

The following notions weak notions are presented in [2] :

- every fuzzy subset $\mu : X \to [0,1]$ such that $\mu \leqslant 1 - \mu$ is called a W-empty set;
- every fuzzy subset $\mu : X \to [0,1]$ such that $\mu \geqslant 1 - \mu$ is called a W-universum;
- every pair of fuzzy subsets (μ, ν) such that $\mu : X \to [0,1]$, $\nu : X \to [0,1]$ and $\mu \leqslant 1 - \nu$ are cal-

led a W-separated sets.
More details about them we can find in [2] . In this paper the theory of fuzzy relations based on weak notions will be presented. Obtained simple results will be used for fuzzy preference relation inreal line interpretated as fuzzy relation "less or equal".

2. Preliminary notions

This section contains a short list of mostly known notions displaced from classical to the fuzzy relation theory. Proposed definitions are based on the weak notions mentioned above. This approach is novel. Therefore, all considerations are presented here.

Let be given the crisp set X . The diagonal of Cartesian product X^2 we defined as the set $D(X) = \{(x,x): x \in X\}$.

Any crisp relation (R) in X is described by the subset $R = \{(x,y): x \varrho y\} \subset X^2$. Its inverse R^{-1} and complement $\overline{R}$ are defined by the subsets $R^{-1} = \{(x,y) : (y,x) \in R\}$ and $\overline{R} = \{(x,y) : (x,y) \notin R\}$.

Usually we mark off the following properties of R:

- reflexivity

$$\forall_{x \in X} \quad x \varrho x \quad ,$$

- antireflexivity

$$\forall_{x \in X} \quad \sim x \varrho x \quad ,$$

- symmetry

$$\forall_{(x,y) \in X^2} \quad x \varrho y \Leftrightarrow y \varrho x \quad ,$$

- antisymmetry

$$\forall_{(x,y) \in X^2} \quad x \varrho y \Rightarrow \sim y \varrho x \quad ,$$

-quasi-antisymmetry

$$\underset{(x,y)\in S^2\subset X^2}{\forall} \quad (x\varrho y \text{ and } y\varrho x) \Rightarrow S^2 \subset D(X) \quad ,$$

- transitivity

$$\underset{(x,y,z)\in X^3}{\forall} \quad x\varrho z \text{ and } z\varrho y \Rightarrow x\varrho y \quad .$$

Transitivity can be defined equivalently as follows

$$\underset{(x,y,z)\in X^3}{\forall} \quad \sim(x\varrho z \text{ and } z\varrho y) \text{ or } x\varrho y \quad .$$

We note that above definitions base on the notions: empty set (antireflexivity), universum (reflexivity, quasi-antisymmetry, transitivity) and separated sets (antisymmetry).

A fuzzy relation (FR) in X is described by fuzzy subset $\varrho : X^2 \rightarrow [0,1]$. Its inverse (FR^{-1}) and complement ($\overline{FR}$) are defined respectively by their membership functions: $\varrho^{-1}(x,y) = \varrho(y,x)$ and $\bar{\varrho}(x,y) = 1 - \varrho(x,y)$. Let us define the diagonal of FR ϱ as the mapping $\delta[\varrho] : X \rightarrow [0,1]$ described by identity $\delta[\varrho](x) = \varrho(x,x)$.

Substituting crisp notions by weak notions we propose to accept the following definitions.

Definition 2.1: An FR ϱ is reflexive iff $\delta[\varrho]$ is a W-universum in X .

Definition 2.2: An FR ϱ is antireflexive iff $\delta[\varrho]$ is a W-empty set in X .

Definition 2.3: An FR ϱ is symmetrical iff $\varrho = \varrho^{-1}$.

Definition 2.4: An FR ϱ is antisymmetrical iff ϱ and ϱ^{-1} are W-separated sets.

Definition 2.5: An FR ϱ is quasi-antisymmetrical iff the condition: " $\varrho \wedge \varrho^{-1}$ is W-universum in $S^2 \subset X^2$ " is sufficient for $S^2 \subset D(X)$.

Definition 2.6: An FR ϱ is transitive iff $(1 - \varrho(\cdot, z) \wedge \wedge \varrho(z, \cdot)) \vee \varrho(\cdot, \cdot)$ is W-universum in X^2 for every $z \in X$.

It is very easy verify that proposed here definitions are more general than one given by Orlovsky [1] . Therefore, the next definitions are more general, too.

Definition 2.7: An FR is called a fuzzy equivalence relation if it is reflexive, symmetrical and transitive.

Definition 2.8: An FR is called a fuzzy strict order relation if it is antisymmetrical and transitive.

Definition 2.9: An FR is called a fuzzy quasi-order relation if it is reflexive and transitive.

3. Fuzzy preference relation

As it is known, unfuzzy preferences are usually modelled by quasi-order relation (PR). It generates equality relation (PR_e) and strict preference relation (PR_s) defined as follows:

$PR_e = PR \cap PR^{-1}$,

$PR_s = PR \cap \overline{PR^{-1}}$.

Let us displace this notions to the theory of fuzzy relations. We shall assume that FR $\varrho : X^2 \to [0,1]$ is specified in the given crisp set X . It will be called a fuzzy preference relation (FPR). Generally, we do not make an assumptions about FPR. This approach differs from one given by Orlovsky, he assumes that FPR is reflexive.

In special case, if the FPR is fuzzy quasi-order relation, we shall say that the FPR is well-defined.

First we define two fuzzy relations corresponding to given FPR: fuzzy equality (FPR_e) and fuzzy strict preference relation (FPR_s) by their membership functions:

$$FPR_e : \quad \varrho_e = \varrho \wedge \varrho^{-1} , \tag{3.1}$$

$$FPR_s : \quad \varrho_s = \varrho \wedge \overline{\varrho^{-1}} . \tag{3.2}$$

Each triplet $(\varrho, \varrho_e, \varrho_s)$ is called a system of fuzzy preferences (SFP) in X generated by FPR.

In general case we obtain the next conclusions.

Lemma 3.1: An FPR_e is symmetrical.

Proof: $\varrho_e^{-1} = \varrho^{-1} \wedge (\varrho^{-1})^{-1} = \varrho^{-1} \wedge \varrho = \varrho_e$. ■

Lemma 3.2: An FPR_s is antisymmetrical.

Proof: $\varrho_s = \varrho \wedge \overline{\varrho^{-1}} \leqslant \overline{\varrho^{-1}} = 1 - \varrho^{-1} \leqslant 1 - \varrho^{-1} \wedge \overline{(\varrho^{-1})^{-1}} =$
$= 1 - \varrho_s^{-1}$. ■

Furthermore, we have:

Theorem 3.1: If the FPR is well-defined then the FPR_e is a fuzzy equivalence relation.

Proof: In general we have

$$\delta[\varrho_e] = \delta[\varrho \wedge \varrho^{-1}] = \delta[\varrho] \wedge \delta[\varrho^{-1}] = \delta[\varrho].$$

So, reflexivity of FPR implies reflexivity of FPR_e.

By transitivity of FPR we obtain

$$(1 - \varrho(x,z) \wedge \varrho(z,y)) \vee \varrho(x,y) \geqslant \frac{1}{2} \qquad (3.3)$$

for every $(x,z,y) \in X^3$. Therefore, we get

$$(1 - \varrho_e(x,z) \wedge \varrho_e(z,y)) \vee \varrho_e(x,y) =$$

$$= (1 - \varrho(x,z) \wedge \varrho(z,x) \wedge \varrho(z,y) \wedge \varrho(y,z)) \vee (\varrho(x,y) \wedge \varrho(y,x)) =$$

$$= (1 - \varrho(x,z) \wedge \varrho(z,y)) \vee (1 - \varrho(z,x) \wedge \varrho(y,z)) \vee (\varrho(x,y) \wedge \varrho(y,x)) =$$

$$= ((1 - \varrho(x,z) \wedge \varrho(z,y)) \vee (1 - \varrho(z,x) \wedge \varrho(y,z)) \vee \varrho(x,y)) \wedge$$
$$\wedge ((1 - \varrho(x,z) \wedge \varrho(z,y)) \vee (1 - \varrho(z,x) \wedge \varrho(y,z)) \vee \varrho(y,x)) \geqslant$$
$$\geqslant ((1 - \varrho(x,z) \wedge \varrho(z,y)) \vee \varrho(x,y)) \wedge ((1 - \varrho(y,z) \wedge \varrho(z,x)) \vee \varrho(y,x)) \geqslant$$
$$\geqslant \frac{1}{2}$$

for every $(x,z,y) \in X^3$. This inequality proves the transitivity of FPR_e. The Lemma 3.1 puts on the end the proof. ■

Theorem 3.2: If the FPR is well-defined then the FPR_s is a fuzzy strict order relation.

Proof: By (3.3) we obtain

$$(1 - \varrho_s(x,z) \wedge \varrho_s(z,y)) \vee \varrho_s(x,y) =$$

$$= (1 - \varrho(x,z) \wedge (1 - \varrho(z,x)) \wedge \varrho(z,y) \wedge (1 - \varrho(y,z))) \vee (\varrho(x,y) \wedge$$
$$\wedge (1 - \varrho(y,x))) = (1 - \varrho(x,z)) \vee \varrho(z,x) \vee (1 - \varrho(z,y)) \vee \varrho(y,z) \vee$$
$$\vee (\varrho(x,y) \wedge (1 - \varrho(y,x)) = ((1 - \varrho(x,z)) \vee \varrho(z,x) \vee (1 - \varrho(z,y)) \vee$$

$\vee \varrho(y,z) \vee \varrho(x,y)) \wedge ((1 - \varrho(x,z)) \vee \varrho(z,x) \vee (1 - \varrho(z,y)) \vee \varrho(y,z) \vee$
$\vee (1 - \varrho(y,x))) \geqslant ((1 - \varrho(x,z)) \vee (1 - \varrho(z,y)) \vee \varrho(x,y)) \wedge (\varrho(z,x) \vee$
$\vee (1 - \varrho(z,y)) \vee (1 - \varrho(y,x))) = ((1 - \varrho(x,z) \wedge \varrho(z,y)) \vee \varrho(x,y)) \wedge$
$\wedge ((1 - \varrho(z,y) \wedge \varrho(y,x)) \vee \varrho(z,x)) \geqslant \frac{1}{2}$

for every $(x,z,y) \in X^3$. This fact along with the Lemma 3.2 proves this theorem. ■

For any fixed element $y \in X$ the mapping $\varrho(y,\cdot)$ defines a fuzzy subset of all elements in X which nonstrictly dominate y . Then the intersection of all fuzzy subsets $\varrho(y,\cdot)$ defined for every $y \in X$, represents the fuzzy subset of those elements in X which nonstrictly dominate all elements in X . We shall call this fuzzy subset the fuzzy nonstrict dominant in X . Thus, according to the definition of the intersection we define its membership function as follows

$$\mu^{NSD}(\cdot) = \inf_{y \in X} \{\varrho(y,\cdot)\} \tag{3.4}$$

The value $\mu^{NSD}(x)$ represents the degree to which the element x nonstrictly dominates all elements in X . If $\mu^{NSD}(x) = 1$ then x will be called unfuzzy nonstrict dominant in X and in this case we shall use the notation

$$X^{UNSD} = \{x:\ x \in X,\ \mu^{NSD}(x) = 1\} \ . \tag{3.5}$$

Obviously, X^{UNSD} can be empty set.

Analogous way, as above, we define the fuzzy nonstrict undominant in X by its membership function

$$\mu^{NSU}(\cdot) = \inf_{y \in X} \{\varrho(\cdot,y)\} \tag{3.6}$$

and unfuzzy nonstrict undominant in X as subset

$$X^{UNSU} = \{ x: x \in X \quad \mu^{NSU}(x) = 1 \} . \tag{3.7}$$

Remark: The following parts of this paper will be published in next Busefal.

References

[1] S.A.Orlovsky, Decision making with a fuzzy preference relation, Fuzzy Sets and Systems 1 (1978), 155-168

[2] K.Piasecki, New concept of separated fuzzy subsets, Proc. Polish Symposium on Interval and Fuzzy Mathematics 1983 /in print/