ON THE REALIZABLE FUZZY SYMMETRIC RELATION AND ITS CONTENT

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ABSTRACT

In the paper [1], LIU Wangjin introduced the concepts of a realizable L-fuzzt symmetric matrix and its content, and obtained the result such that a fuzzy symmetric matrix $B\in L^{2\times 2}$ is realizable iff $B=B^2$. In this paper, we give the definition of a realizable fuzzy symmetric relation as the generalization of problem in [1], and show a necussary and sufficient condition for a realizable fuzzy symmetric relation. We also obtain some results about content.

1. INTRODUCTION

Let X and Y be two nonempty sets.

Definition 1.1. Let $\underset{\sim}{R}^{(\gamma)}$, $\gamma\in\Gamma$ be a family of fuzzy relation on X×Y. We call that the fuzzy relation on X×Y

$$\underset{\sim}{R} = \bigsqcup_{\gamma\in\Gamma} \underset{\sim}{R}^{(\gamma)},$$

determined by

$$\underset{\sim}{R}(x,y) = \bigvee_{\gamma\in\Gamma} \underset{\sim}{R}^{(\gamma)}(x,y), \qquad \forall x\in X,\ \forall y\in Y,$$

the sum of $\underset{\sim}{R}^{(\gamma)}$, $\gamma\in\Gamma$.

Definition 1.2. Let $\underset{\sim}{R}$ be a fuzzy relation on X×Y, and $\underset{\sim}{S}$ be a fuzzy relation on Y×Z. We call the fuzzy relation on X×Z

$$\underset{\sim}{T} = \bigsqcup_{\lambda\in[0,1]} \lambda(\underset{\sim}{R}_\lambda * \underset{\sim}{S}_\lambda)$$

the composition Of $\underset{\sim}{R}$ and $\underset{\sim}{S}$, and denote it by

$$\underset{\sim}{T} = \underset{\sim}{R} * \underset{\sim}{S},$$

where $\underset{\sim}{R}_\lambda = \{(x,y) \mid \underset{\sim}{R}(x,y)\geq\lambda\}$, $\underset{\sim}{S}_\lambda = \{(y,z) \mid \underset{\sim}{S}(y,z)\geq\lambda\}$.

We have

$$(\underset{\sim}{R}*\underset{\sim}{S})(x,z) = \bigvee_{y\in Y} [\underset{\sim}{R}(x,y) \wedge \underset{\sim}{S}(y,z)], \quad \forall x \in X, \forall z \in Z.$$

<u>Definition 1.3.</u> Let $\underset{\sim}{R}$ be a fuzzy relation on $X\times Y$. We call the fuzzy relation $\underset{\sim}{R}'$ on $Y\times X$ the transposed relation of $\underset{\sim}{R}$, iff

$$\underset{\sim}{R}'(y,x) = \underset{\sim}{R}(x,y), \quad \forall y \in Y, \forall x \in X.$$

<u>Definition 1.4.</u> Let $\underset{\sim}{R}$ be a fuzzy relation on $X\times X$. We call $\underset{\sim}{R}$ a fuzzy symmetric relation on X iff

$$\underset{\sim}{R}(x,x') = \underset{\sim}{R}'(x,x'), \quad \forall x,x' \in X.$$

It is easy to see, that $\underset{\sim}{R}' = \underset{\sim}{R}$ iff

$$\underset{\sim}{R}(x,x') = \underset{\sim}{R}(x',x), \quad \forall x, x' \in X,$$

where $\underset{\sim}{R}$ is a fuzzy relation on $X\times X$.

<u>Theorem 1.1.</u> Let $\underset{\sim}{R}$ be a fuzzy relation on $X\times Y$. Then $\underset{\sim}{R} * \underset{\sim}{R}'$ is a fuzzy symmetric relation on X.

<u>Proof:</u>

$$\begin{aligned}(\underset{\sim}{R}*\underset{\sim}{R}')(x,x') &= \bigvee_{y\in Y} [\underset{\sim}{R}(x,y) \wedge \underset{\sim}{R}'(y,x')] \\ &= \bigvee_{y\in Y} [\underset{\sim}{R}(x,y) \wedge \underset{\sim}{R}(x',y)] \\ &= \bigvee_{y\in Y} [\underset{\sim}{R}(x',y) \wedge \underset{\sim}{R}'(y,x)] \\ &= (\underset{\sim}{R}*\underset{\sim}{R}')(x',x), \quad \forall x,x' \in X.\end{aligned}$$

∎

2. THE REALIZABLE PROBLEM FOR FUZZY SYMMETRIC RELATION AND ITS CONTENT

<u>Definition 2.1.</u> Let $\underset{\sim}{B}$ be a fuzzy symmetric relation on X. If there are a set Y and a fuzzy relation $\underset{\sim}{R}$ on $X\times Y$, such that

$$\underset{\sim}{B} = \underset{\sim}{R} * \underset{\sim}{R}',$$

then $\underset{\sim}{B}$ is called a realizable fuzzy symmetric relation on X, and $\underset{\sim}{R}$ a realization of $\underset{\sim}{B}$, and

$$\gamma(\underset{\sim}{B}) = \inf_{Y} \{ \text{car}Y \mid \exists \underset{\sim}{R} \text{ on } X \times Y, \underset{\sim}{B} = \underset{\sim}{R} * \underset{\sim}{R}' \}$$

the content of $\underset{\sim}{B}$, where car$Y$ is the cardinal numbers of Y.

Theorem 2.1. Let $\underset{\sim}{B}$ be a realizable fuzzy symmetric relation on X, then

$$\underset{\sim}{B}(x,x') \leq \underset{\sim}{B}(x,x), \qquad \forall x,\ x' \in X. \tag{1}$$

Proof: Suppose $\underset{\sim}{B} = \underset{\sim}{R} * \underset{\sim}{R}'$, where $\underset{\sim}{R}$ is a fuzzy relation on $X \times Y$, Y is a nonempty set. Then

$$\begin{aligned} \underset{\sim}{B}(x,x') &= \bigvee_{y \in Y} [\underset{\sim}{R}(x,y) \wedge \underset{\sim}{R}(x',y)] \\ &\leq \bigvee_{y \in Y} [\underset{\sim}{R}(x,y) \wedge \underset{\sim}{R}(x,y)] \\ &= \bigvee_{y \in Y} [\underset{\sim}{R}(x,y) \wedge \underset{\sim}{R}'(y,x)] \\ &= \underset{\sim}{B}(x,x). \end{aligned}$$

∎

If $\underset{\sim}{B}$ satisfies (1), we say that $\underset{\sim}{B}$ is quasireflexive.

Definition 2.2. Let $\underset{\sim}{R}$ be a fuzzy relation on $X \times Y$, and X' be a nonempty subset of X. We call the fuzzy relation $\underset{\sim}{R}_{X'}$ on $X \times Y$ the single limitation relation of $\underset{\sim}{R}$ on X', if

$$\underset{\sim}{R}_{X'}(x,y) = \underset{\sim}{R}(x,y), \qquad \forall x \in X',\ \forall y \in Y.$$

Definition 2.3. Let $\underset{\sim}{B}$ be a fuzzy relation on $X \times X$, and X' be a nonempty subset of X. We call the fuzzy relation $\underset{\sim}{B}_{2X'}$ on $X' \times X'$ the double limitation relation of $\underset{\sim}{B}$ on X, if

$$\underset{\sim}{B}_{2X'}(x,x') = \underset{\sim}{B}(x,x'), \qquad \forall x,\ x' \in X.$$

Theorem 2.2. Let $\underset{\sim}{B}$ be a realizable fuzzy symmetric relation on X. Then $\underset{\sim}{B}_{2X'}$ is a realizable symmetric relation on X', where X' is a nonempty subset of X.

Proof: Suppose $\underset{\sim}{B} = \underset{\sim}{R} * \underset{\sim}{R}'$, where $\underset{\sim}{R}$ is a fuzzy relation on $X \times X$, Y is a nonempty set. We have

$$\underset{\sim}{B}_{2X'} = \underset{\sim}{R}_{X'} * (\underset{\sim}{R}_{X'})'.$$

Indeed,

$$[\underset{\sim}{R}_{x'} * (\underset{\sim}{R}_{x'})'](x, x')$$
$$= \bigvee_{y \in Y} [\underset{\sim}{R}_{x'}(x, y) \wedge \underset{\sim}{R}_{x'}(x', y)]$$
$$= \bigvee_{y \in Y} [\underset{\sim}{R}(x, y) \wedge R(x', y)]$$
$$= \underset{\sim}{B}(x, x') = \underset{\sim}{B}_{2x'}(x, x'), \qquad \forall x, x' \in X'. \qquad \blacksquare$$

Theorem 2.3. Let $\underset{\sim}{B}^{(\gamma)}$, $\gamma \in \Gamma$ be a family of realizable fuzzy symmetric relations on X. Then

$$\underset{\sim}{B} = \bigcup_{\gamma \in \Gamma} \underset{\sim}{B}^{(\gamma)}$$

is a realizable fuzzy symmetric relation on X.

Proof: Suppose $\underset{\sim}{B}^{(\gamma)} = \underset{\sim}{A}^{(\gamma)} * \underset{\sim}{A}^{(\gamma)'}$, where $\underset{\sim}{A}^{(\gamma)}$ is a fuzzy relation on $X \times Y^{(\gamma)}$. We may suppose that $Y^{(\gamma)}$, $\gamma \in \Gamma$ is disjoint each other.

Consider $Y = \bigcup_{\gamma \in \Gamma} Y^{(\gamma)}$, and the fuzzy relation $\underset{\sim}{R}^{(\gamma)}$ on $X \times Y$, which is defined by

$$\underset{\sim}{R}^{(\gamma)}(x, y) = \begin{cases} \underset{\sim}{A}^{(\gamma)}(x, y), & \text{if } y \in Y^{(\gamma)}, \\ 0, & \text{if } y \notin Y^{(\gamma)}. \end{cases}$$

Take a fuzzy relation $\underset{\sim}{R}$ on $X \times Y$,

$$\underset{\sim}{R} = \bigcup_{\gamma \in \Gamma} R^{(\gamma)},$$

we show that

$$\underset{\sim}{B} = \underset{\sim}{R} * \underset{\sim}{R}'.$$

In fact, $\quad \underset{\sim}{B}(x, x') = (\bigcup_{\gamma \in \Gamma} \underset{\sim}{B}^{(\gamma)})(x, x')$

$$= \bigvee_{\gamma \in \Gamma} \underset{\sim}{B}^{(\gamma)}(x, x')$$
$$= \bigvee_{\gamma \in \Gamma} [\bigvee_{y \in Y^{(\gamma)}} (\underset{\sim}{A}^{(\gamma)}(x, y) \wedge \underset{\sim}{A}^{(\gamma)}(x', y))]$$
$$= \bigvee_{y \in Y} [\bigvee_{\gamma \in \Gamma} (\underset{\sim}{R}^{(\gamma)}(x, y) \wedge \underset{\sim}{R}^{(\gamma)}(x', y))].$$

Since $Y^{(\gamma)}$ is disjoint each other for $\gamma \in \Gamma$, there is a unique $\gamma_0 \in \Gamma$ for any $y \in Y$, such that

$$\underset{\sim}{R}^{(\gamma)}(x, y) = 0, \quad \forall x \in X, \quad \gamma \neq \gamma_0.$$

Then

$$\bigvee_{\gamma\in\Gamma}(\underset{\sim}{R}^{(\gamma)}(x,y)\wedge\underset{\sim}{R}^{(\gamma)}(x',y)) = (\bigvee_{\gamma\in\Gamma}\underset{\sim}{R}^{(\gamma)}(x,y))\wedge(\bigvee_{\gamma\in\Gamma}\underset{\sim}{R}^{(\gamma)}(x',y)), \quad \forall y\in Y.$$

Therefore,

$$\begin{aligned}\underset{\sim}{B}(x,x') &= \bigvee_{y\in Y}\left[(\bigvee_{\gamma\in\Gamma}\underset{\sim}{R}^{(\gamma)}(x,y))\wedge(\bigvee_{\gamma\in\Gamma}\underset{\sim}{R}^{(\gamma)}(x',y))\right]\\ &= \bigvee_{y\in Y}\left[\underset{\sim}{R}(x,y)\wedge\underset{\sim}{R}(x',y)\right]\\ &= (\underset{\sim}{R} * \underset{\sim}{R}')(x,x'), \qquad \forall x,\ x'\in X. \end{aligned}$$ ∎

The following theorem is the main result in this paper.

Theorem 2.4. Let $\underset{\sim}{B}$ be a fuzzy symmetric relation on X. Then $\underset{\sim}{B}$ is realizable iff it is quasireflexive.

Proof: The necessity is given by theorem 2.1.

To prove the sufficiency, suppose that $\underset{\sim}{B}$ satisfies the condition (1). We denote

$$\underset{\sim}{B}_{xx'}(a,b) = \begin{cases} \underset{\sim}{B}(a,b), & \text{if } a,\ b\in\{x,x\}\\ 0, & \text{if } a\notin\{x,\ x\} \text{ or } b\notin\{x,\ x\},\end{cases}$$

for any $x,\ x'\in X,\ x\neq x'$. It is easy to verify that $\underset{\sim}{B}_{x,x'}$ is realizable. Since

$$\underset{\sim}{B} = \bigcup_{x,x'\in X}\underset{\sim}{B}_{xx'},$$

then $\underset{\sim}{B}$ is realizable, by theorem 2.3. ∎

3. SOME RESULTS ABOUT CONTENT

Theorem 3.1. Let X be a nonempty set, the cardinal number of X be a finite n, and $\underset{\sim}{B}$ be a realizable fuzzy symmetric relation on X. Then

$$\gamma(B)\leq n(n-1).$$

Proof: By the proofs of theorem 2.3, 2.4, we have

$$\gamma(B)\leq N,$$

where N is the cardinal numbers of Y, Y is the set consisting of every subset which contains any two elements in X. We have N=n(n-1). Therefore,

$\gamma(B) \leqq n(n-1)$. ▮

Theorem 3.2. Let X be a nonempty set, with the cardinal number $c \geq \aleph_0$, and $\underset{\sim}{B}$ be a realizable fuzzy symmetric relation on X. Then

$\gamma(B) \leqq c$. ▮

Theorem 3.3. Let X be a nonempty set, and $\underset{\sim}{B}$ be a realizable fuzzy symmetric relation on X. Then $\gamma(B) = 1$, iff

$$\underset{\sim}{B}(x,x') = \underset{\sim}{B}(x,x)\wedge\underset{\sim}{B}(x',x'), \qquad \forall x,\ x' \in X.$$

Proof: Since $\gamma(B) = 1$, there are a set Y which only contains a element and a fuzzy relation R on X×Y such that

$$\underset{\sim}{B} = \underset{\sim}{R} * \underset{\sim}{R}'.$$

Thus,

$$\underset{\sim}{B}(x,x')=\underset{\sim}{R}(x,y)\wedge\underset{\sim}{R}'(y,x')=\underset{\sim}{R}(x,y)\wedge\underset{\sim}{R}(x',y),$$

$$\underset{\sim}{B}(x,x)=\underset{\sim}{R}(x,y)\wedge\underset{\sim}{R}(x,y)=\underset{\sim}{R}(x,y),$$

$$\underset{\sim}{B}(x',x')=\underset{\sim}{R}(x',y)\wedge\underset{\sim}{R}(x',y)=\underset{\sim}{R}(x',y).$$

Then

$$\underset{\sim}{B}(x,x')=\underset{\sim}{B}(x,x)\wedge\underset{\sim}{B}(x',x').$$

Conversely, if

$$\underset{\sim}{B}(x,x')=\underset{\sim}{B}(x,x)\wedge\underset{\sim}{B}(x',x'), \qquad x,x' \quad X,$$

we take

$$\underset{\sim}{R}(x,y)=\underset{\sim}{B}(x,x), \qquad \forall x\in X.$$

It is a fuzzy relation on X×Y, where $Y = \{y\}$. Then

$$\underset{\sim}{B} = \underset{\sim}{R} * \underset{\sim}{R}'.$$

Therefore,

$\gamma(B) = 1$. ▮

REFERENCES

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[2] Dubois, D. and Prade, H., Fuzzy Sets and Systems —— theory and applications, Academic Press, New York, 1980.