# RANDOM SETS REPRESENTATION OF FUZZY SETS' COUNTABLE OPERATIONS

Yang Jian Li , Fu Xiang

Dept of Math, Beijing Normal University, Beijing China.

Wang Pei-Zhuang, I.R.Goodman and others have related fuzzy set to random set and constructed the structure of falling space (see [1], [2], [3]). In this paper we put forward the concept of random sets delegation with respect to fuzzy sets' countable union, imtersection and complement defined by Zadeh in [7]. On the basis of [1], we go further into the problem about the relationship between the fallability and the measurability of fuzzy set.

## §1. Introduction

Let $U$ be universal, $\mathcal{F}(U)$ be the set of fuzzy subsets of U, $\mathcal{B}$ be $\sigma$-field on U satisfying $\forall u \in U \Longrightarrow \{u\} \in \mathcal{B}$. For each $u \in U$, define

$$\dot{u} = \{ B \in \mathcal{B} \mid u \in B \} \tag{1.1}$$

Let $\check{\mathcal{B}}$ be $\sigma$-field generated by $\{\dot{u} \mid u \in U\}$.

Definition1.1 Put $\mathcal{P}(U) = \{A \mid A \subset U\}$, map $\xi: \Omega \to \mathcal{P}(U)$ is called set-valued map from $\Omega$ to $U$. Let $\xi$, $\xi_t$, $t \in T$ be set-valued maps, $\bigcap_{t\in T} \xi_t$, $\bigcup_{t\in T} \xi_t$, $\xi^c$ are defined as follows:

$$\left(\bigcap_{t\in T} \xi_t\right)(\omega) = \bigcap_{t\in T} \xi_t(\omega), \quad \forall \omega \in \Omega \tag{1.2}$$

$$\left(\bigcup_{t\in T} \xi_t\right)(\omega) = \bigcup_{t\in T} \xi_t(\omega), \quad \forall \omega \in \Omega \tag{1.3}$$

$$(\xi^c)(\omega) = (\xi(\omega))^c, \quad \forall \omega \in \Omega \tag{1.4}$$

Definition1.2 Let $(\Omega, \mathcal{A}, P)$ be a probability space, we say $(\Omega, \mathcal{A}, P; U, \mathcal{B}, \check{\mathcal{B}})$ is fall-shadow space. Map $\xi: \Omega \to \mathcal{B}$ is called random set if $\xi$ is $\mathcal{A}$-$\check{\mathcal{B}}$ measurable. $\mathcal{S}$ denotes the collection of all random sets. For every $\xi \in \mathcal{S}$, the fuzzy set

$$\mu_\xi(u) = P(\xi^{-1}(\dot{u})), \quad \forall u \in U \tag{1.5}$$

is called the fuzzy fall shadow of $\xi$. Fuzzy set $\underset{\sim}{A}$ over U is called measurable if $\underset{\sim}{A}$ is $\mathscr{B}$-measurable function; $\underset{\sim}{A}$ is called fallable if there exists a random set $\xi$ such that $\mu_\xi = \underset{\sim}{A}$. All measurable and all fallable fuzzy sets over U are respectively denoted by $\mathscr{F}_0(U)$ and $\mathscr{F}_1(U)$. If $\mathscr{F}_0(U)=\mathscr{F}_1(U)$, we say $(\Omega, \mathscr{A}, P; U, \mathscr{B}, \hat{\mathscr{B}})$ is complete falling measurable structure.

Proposition1.1 If T is countable, the $\mathscr{T}$ is closed under $\bigcap_{t\in T}$, $\bigcup_{t\in T}$, $^c$. ▯

Let $(\Omega, \mathscr{A}, m)$ be any finite measure space, for $\forall \varepsilon>0$, we introduce inclusion relation:

$$A \underset{m-\varepsilon}{\subseteq} B \ (\text{or } B \underset{m-\varepsilon}{\supseteq} A) \overset{\Delta}{\Longleftrightarrow} \text{there is } C \in \mathscr{A}, \text{ such that } m(C) \le \varepsilon \text{ and } A \setminus C \subset B \tag{1.6}$$

Definition1.3 $A_n \in \mathscr{A}$, n=1,2,..., $\{A_n\}$ is called m-intersction sequence (or m-union sequence), if for $\forall$ i,j and $\forall \varepsilon>0$, there is some $k$ such that

$$A_k \underset{m-\varepsilon}{\subseteq} A_i \cap A_j \quad (\text{or } A_i \cup A_j \underset{m-\varepsilon}{\subseteq} A_k) \tag{1.7}$$

Proposition1.2 $\{A_n\}$ is m-intersction sequence iff $\{A_n^c\}$ is m-union sequence. ▯

Proposition1.3 If $\{A_n\} \subset \mathscr{A}$, then

(i) $m(\bigcap_{n=1}^{\infty} A_n) = \inf_n m(A_n)$ iff $\{A_n\}$ is m-intersection sequence.

(ii) $m(\bigcup_{n=1}^{\infty} A_n) = \sup_n m(A_n)$ iff $\{A_n\}$ is m-union sequence. ▯

Corollary1.1 If $A_i \in \mathscr{A}$, i=1,...,n. then

(i) $m(\bigcap_{i=1}^{n} A_i) = \min_{1\le i\le n} \{m(A_i)\}$ iff there is i, $1\le i\le n$ such that $A_i \underset{m-0}{\subseteq} A_j$, j=1,...,n.

(ii) $m(\bigcup_{i=1}^{n} A_i) = \max_{1\le i\le n} \{m(A_i)\}$ iff there is i, $1\le i\le n$ such that $A_j \underset{m-0}{\subseteq} A_i$, j=1,...n.

## §2. Delegation on operations of fuzzy sets

Definition2.1 Suppose $(\Omega, \mathscr{A}, P; U, \mathscr{B}, \hat{\mathscr{B}})$ be a fall-shadow space, and $\mathscr{F}_1(U)$ is closed under countable intersection, union and comlement defined by L.A.Zadeh in[7]. $\mathscr{D} = \{ \xi_{\underset{\sim}{A}} \mid \underset{\sim}{A} \in \mathscr{F}_1(U), \mu_{\xi_{\underset{\sim}{A}}} = \underset{\sim}{A} \}$ is called delegation with respect to fuzzy sets' countable intersection, countable union and complement (delegation for short), if any $\xi_{\underset{\sim}{A}}$ and $\xi_{\underset{\sim}{A}_n}$ (n=1,2,...) $\in \mathscr{D}$, hold

$$\mu_{\bigcap_{n=1}^{\infty} \xi_{\underset{\sim}{A}_n}} = \bigcap_{n=1}^{\infty} \mu_{\xi_{\underset{\sim}{A}_n}} = \bigcap_{n=1}^{\infty} \underset{\sim}{A}_n \tag{2.1}$$

$$\mathcal{M}_{\bigcup\limits_{n=1}^{\infty} \xi_{\underset{\sim}{A}_n}} = \bigcup_{n=1}^{\infty} \mathcal{M}_{\xi_{\underset{\sim}{A}_n}} = \bigcup_{n=1}^{\infty} \underset{\sim}{A}_n \tag{2.2}$$

$$\mathcal{M}_{(\xi_{\underset{\sim}{A}})^c} = (\mathcal{M}_{\xi_{\underset{\sim}{A}}})^c = \underset{\sim}{A}^c \tag{2.3}$$

Proposition2.1 Let $(\Omega, \mathcal{A}, P; U, \mathcal{B}, \check{\mathcal{B}})$ be fall-shadow space, $\{\xi_n\}_{n=1,2,\ldots} \subset \mathcal{J}$ ,then

(i) $\bigcap\limits_{n=1}^{\infty} \mathcal{M}_{\xi_n} = \mathcal{M}_{(\bigcap\limits_{n=1}^{\infty} \xi_n)}$ iff $\forall u \in U$, $\{\xi_n^{-1}(\dot{u})\}$is P-intersection sequence .

(ii) $\bigcup\limits_{n=1}^{\infty} \mathcal{M}_{\xi_n} = \mathcal{M}_{(\bigcup\limits_{n=1}^{\infty} \xi_n)}$ iff $\forall u \in U$, $\{\xi_n^{-1}(\dot{u})\}$is P-union sequence.

Proof. $\bigcap\limits_{n=1}^{\infty} \mathcal{M}_{\xi_n} = \mathcal{M}_{(\bigcap\limits_{n=1}^{\infty} \xi_n)} \Longleftrightarrow \forall\, u \in U, \inf\limits_n \mathcal{M}_{\xi_n}(u) = \mathcal{M}_{(\bigcap\limits_{n=1}^{\infty} \xi_n)}(u) \Longleftrightarrow$ $\forall u$, $\inf\limits_n P(\xi_n^{-1}(\dot{u})) = P((\bigcap\limits_{n=1}^{\infty} \xi_n)^{-1}(\dot{u})) \Longleftrightarrow \forall u, \inf\limits_n P(\xi_n^{-1}(\dot{u})) = P(\bigcap\limits_{n=1}^{\infty} \xi_n^{-1}(\dot{u}))$

From Proposition1.3 (i)is clear. We can prove (ii) in the similar way. $\square$

Proposition2.2 For any random set $\xi$ ,it holds $\mathcal{M}_{\xi^c} = (\mathcal{M}_{\xi})^c$. Therefor $\mathcal{D}$ is delegation as long as satisfy (2.1),(2.2). $\square$

Theorem2.1 $\mathcal{D} = \{\xi_{\underset{\sim}{A}} \mid \underset{\sim}{A} \in \mathcal{F}_1(U), \mathcal{M}_{\xi_{\underset{\sim}{A}}} = \underset{\sim}{A}\}$ is delegation iff $\forall u \in U$, $\{\xi_{\underset{\sim}{A}}^{-1}(\dot{u}) \mid \underset{\sim}{A} \in \mathcal{F}_1(U)\}$ is linear order set according to $\underset{P\text{-}o}{\subseteq}$.

Proof.Necessary. for any $\xi_{\underset{\sim}{A}}$ , $\xi_{\underset{\sim}{B}} \in \mathcal{D}$, $\mathcal{M}_{(\xi_{\underset{\sim}{A}} \cap \xi_{\underset{\sim}{B}})} = \underset{\sim}{A} \cap \underset{\sim}{B}$ , then for any $u \in U$ holds $P(\xi_{\underset{\sim}{A}}^{-1}(\dot{u}) \cap \xi_{\underset{\sim}{B}}^{-1}(\dot{u})) = \min\{P(\xi_{\underset{\sim}{A}}^{-1}(\dot{u})), P(\xi_{\underset{\sim}{B}}^{-1}(\dot{u}))\}$ from corollary1.1,we assert $\xi_{\underset{\sim}{A}}^{-1}(\dot{u}) \underset{P\text{-}o}{\subseteq} \xi_{\underset{\sim}{B}}^{-1}(u)$ or $\xi_{\underset{\sim}{A}}^{-1}(\dot{u}) \underset{P\text{-}o}{\supseteq} \xi_{\underset{\sim}{B}}^{-1}(\dot{u})$.

Sufficiency. Let $\{\xi_{\underset{\sim}{A}_n}\} \subset \mathcal{D}$ ,for any $\xi_{\underset{\sim}{A}_i}$, $\xi_{\underset{\sim}{A}_j}$, $\forall u$ , we may as well assume $\xi_{\underset{\sim}{A}_i}^{-1}(\dot{u}) \underset{P\text{-}o}{\subseteq} \xi_{\underset{\sim}{A}_j}^{-1}(\dot{u})$, then $\xi_{\underset{\sim}{A}_i}^{-1}(\dot{u}) \underset{P\text{-}o}{\subseteq} \xi_{\underset{\sim}{A}_i}^{-1}(\dot{u}) \cap \xi_{\underset{\sim}{A}_j}^{-1}(\dot{u})$, $\xi_{\underset{\sim}{A}_i}^{-1}(\dot{u}) \cup \xi_{\underset{\sim}{A}_j}^{-1}(\dot{u}) \underset{P\text{-}o}{\subseteq} \xi_{\underset{\sim}{A}_j}^{-1}(\dot{u})$. From proposition2.1 we know(2.1) (2.2) are satisfied. $\square$

IN accordance with the course of the proof we have

Theorem2.2 $\mathcal{D} = \{\xi_{\underset{\sim}{A}} \mid \underset{\sim}{A} \in \mathcal{F}_1(U), \mathcal{M}_{\xi_{\underset{\sim}{A}}} = \underset{\sim}{A}\}$ is delegation iff for any $\xi_{\underset{\sim}{A}}$ , $\xi_{\underset{\sim}{B}} \in \mathcal{D}$, $\mathcal{M}_{(\xi_{\underset{\sim}{A}} \cap \xi_{\underset{\sim}{B}})} = \underset{\sim}{A} \cap \underset{\sim}{B}$. (or equivalentto $\mathcal{M}_{(\xi_{\underset{\sim}{A}} \cup \xi_{\underset{\sim}{B}})} = \underset{\sim}{A} \cup \underset{\sim}{B}$)

## §3. Existence theorem

Definition3.1 Let $(\Omega, \mathcal{A}, P)$ be a probability space,

(i) $\mathcal{J} \subset \mathcal{A}$ is called strict nest of $(\Omega, \mathcal{A}, P)$, if

$((\forall T_1, T_2 \in \mathcal{T}) \Longrightarrow ((T_1 \subset T \text{ or } T_1 \supset T_2) \text{ and } (T_1 \neq T_2 \Rightarrow P(T_1) \neq P(T_2)))$

(ii) $\{A_\lambda\}_{\lambda\in[0,1]} \subset \mathcal{A}$ is called regular nest of $(\Omega, \mathcal{A}, P)$ if

1°. $P(A_\lambda)=\lambda, \quad \forall\lambda\in[0,1]$;

2°. $A_1=\Omega, \ A_0=\phi$;

3°. $(\forall\lambda_1, \lambda_2 \in[0,1])(\lambda_1<\lambda_2 \Longrightarrow A_{\lambda_1} \subset A_{\lambda_2})$.

(iii) $D^*=\{d^{(n)}\}$ a $\mathcal{A}$ partition sequence of $\Omega$ is called regular net of $(\Omega, \mathcal{A}, P)$ if

$$d^{(n)}=\{D_{i_1\cdots i_n} \mid i_k=1,2;\ k=1,\dots,n\}$$
$$D_{i_1\cdots i_n}\in\mathcal{A} \quad (i=1,2;\ k=1,\dots,n) \tag{3.1}$$
$$P(D_{i_1\cdots i_n})=1/2^n,$$
$$D_{i_1\cdots i_n 1}\cup D_{i_1\cdots i_n 2}=D_{i_1\cdots i_n}, \quad n=1,2,\dots.$$

The regular nest is obviousely strict nest.

Lemma3.1 Let $(\Omega, \mathcal{A}, P)$ be a probability space, if there is subset family $\{B_\lambda\}_{\lambda\in[0,1]}$ in $\mathcal{A}$ which satisfys:

(i) $P(B_\lambda)=\lambda \qquad \forall\lambda\in[0,1]$

(ii) $\lambda_1<\lambda_2 \Longrightarrow B_{\lambda_1} \underset{P-0}{\subseteq} B_{\lambda_2} \qquad \forall\lambda_1,\lambda_2\in[0,1]$

then $(\Omega, \mathcal{A}, P)$ has regular nest. The family satisfying (i), (ii) is called quasi-regular nest. ▯

Lemma3.2 $(\Omega, \mathcal{A}, P)$ has regular nest $\Longrightarrow$ $(\Omega, \mathcal{A}, P)$ has regular net.

Proof. For any n-ary repeated permutation composed of 1 or 2 $i_1\cdots i_n$, define

$f(i_1\cdots i_n)$=the frequency of 2's occurance in $i_1\cdots i_n$; (3.2)

when $f(i_1\cdots i_n)\neq 0$, we define

$g_j(i_1\cdots i_n)$=the place figure of jth 2 in $i_1\cdots i_n$ counting from left to right; $j=1,\dots,f(i_1\cdots i_n)$ (3.3)

$g_0(i_1\cdots i_n)=0$;

$$\sigma(i_1\cdots i_n)=\begin{cases}0, & \text{when } f(i_1\cdots i_n)=0;\\ \displaystyle\sum_{j=1}^{g(i_1\cdots i_n)}\frac{1}{2^{g_j(i_1\cdots i_n)}}, & \text{otherwise.}\end{cases}\tag{3.4}$$

Let $D_1=A_{\frac{1}{2}}$, $D_2=\Omega\setminus A_{\frac{1}{2}}$ . Suppose $D_{i_1\cdots i_n}$ has been defined,we construct $D_{i_1\cdots i_n1}=D_{i_1\cdots i_n}\cap A_{\sigma(i_1\cdots i_n)+\frac{1}{2^{n+1}}}$, $D_{i_1\cdots i_n2}=D_{i_1\cdots i_n}\setminus D_{i_1\cdots i_n1}$

In order to show $\{D_{i_1\cdots i_n}\}$ constructed as above is regular nest it is need only to prove that for any n-ary repeatea permutation $i_1\cdots i_n$ consisting of 1 or 2, we have

(1) $D_{i_1\cdots i_n}\cap A_{\sigma(i_1\cdots i_n)}=\Phi$ ;

(2) $A_{\sigma(i_1\cdots i_n)+\frac{1}{2^n}}\subset A_{\sigma(i_1\cdots i_n)}\cup D_{i_1\cdots i_n}$ .

When n=1 , (1), (2) is easy to test.After assuming that for n-1 (1),(2) is true,we consider any n-ary repeated permutation $i_1\cdots i_n$.

When $i_n=1$, $D_{i_1\cdots i_{n-1}1}=D_{i_1\cdots i_{n-1}}\cap A_{\sigma(i_1\cdots i_{n-1})+\frac{1}{2^n}}$, $\sigma(i_1\cdots i_{n-1}1)=\sigma(i_1\cdots i_{n-1})$

$A_{\sigma(i_1\cdots i_{n-1}1)}=A_{\sigma(i_1\cdots i_{n-1})}$, so $D_{i_1\cdots i_n1}\cap A_{\sigma(i_1\cdots i_{n-1}1)}=\Phi$;

When $i_n=2$, $D_{i_1\cdots i_{n-1}2}=D_{i_1\cdots i_{n-1}}\setminus D_{i_1\cdots i_{n-1}1}=D_{i_1\cdots i_{n-1}}\setminus A_{\sigma(i_1\cdots i_{n-1})+\frac{1}{2^n}}$

$\sigma(i_1\cdots i_{n-1}2)=\sigma(i_1\cdots i_{n-1})+\frac{1}{2^n}$, then $A_{\sigma(i_1\cdots i_{n-1}2)}=A_{\sigma(i_1\cdots i_{n-1})+\frac{1}{2^n}}$

therefor $A_{\sigma(i_1\cdots i_{n-1}2)}\cap D_{i_1\cdots i_{n-1}2}=\phi$ ,so (1) is teue.We hope to prove (2).

When $i_n=1$, $D_{i_1\cdots i_{n-1}1}=D_{i_1\cdots i_{n-1}}\cap A_{\sigma(i_1\cdots i_{n-1})+\frac{1}{2^n}}$,

$A_{\sigma(i_1\cdots i_{n-1}1)+\frac{1}{2^n}}=A_{\sigma(i_1\cdots i_{n-1})+\frac{1}{2^n}}\subset A_{\sigma(i_1\cdots i_{n-1})+\frac{1}{2^{n-1}}}$ (with inductive hypothesis)

$\subset A_{\sigma(i_1\cdots i_{n-1})}\cup D_{i_1\cdots i_{n-1}}$ ,then $A_{\sigma(i_1\cdots i_{n-1}1)+\frac{1}{2^n}}\setminus A_{\sigma(i_1\cdots i_{n-1})}\subset D_{i_1\cdots i_{n-1}}$ ,then

$A_{\sigma(i_1\cdots i_{n-1}1)+\frac{1}{2^n}}\setminus A_{\sigma(i_1\cdots i_{n-1}1)}\subset D_{i_1\cdots i_{n-1}1}$ ,from (1)we know

When $i_n=2$, $D_{i_1\cdots i_{n-1}2}=D_{i_1\cdots i_{n-1}}\setminus A_{\sigma(i_1\cdots i_{n-1})+\frac{1}{2^n}}$, $A_{\sigma(i_1\cdots i_{n-1})+\frac{1}{2^n}}=A_{\sigma(i_1\cdots i_{n-1})+\frac{1}{2^n}+\frac{1}{2^n}}$

$=A_{\sigma(i_1\cdots i_{n-1})+\frac{1}{2^{n-1}}}$(with inductive hypothesis)$\subset A_{\sigma(i_1\cdots i_{n-1})}\cup D_{i_1\cdots i_{n-1}}$

$\subset A_{\sigma(i_1\cdots i_{n-1})+\frac{1}{2^n}}\cup D_{i_1\cdots i_{n-1}}=A_{\sigma(i_1\cdots i_{n-1})+\frac{1}{2^n}}\cup D_{i_1\cdots i_n}=A_{\sigma(i_1\cdots i_n)}\cup D_{i_1\cdots i_n}$.

As a result (2) is true. $\square$

<u>Theorem3.1</u> A complete falling measurable structure$(\Omega,\mathcal{A},P;U,\mathcal{B},\mathcal{B})$ has delegation iff $(\Omega,\mathcal{A},P)$ has regular nest.

Proof. Necessary. Suppose $\mathcal{D}=\{\mathfrak{z}_{\underset{\sim}{A}}\mid \underset{\sim}{A}\in\mathcal{F}_1(U),\ \mu_{\mathfrak{z}_{\underset{\sim}{A}}}=\underset{\sim}{A}\}$ is delegation,then $\forall$ u∈U, $\{\mathfrak{z}_{\underset{\sim}{A}}^{-1}(\dot{u})\}$ is a quasi-regular nest,fome which we can construct a regular nest of $(\Omega,\mathcal{A},P)$ by useing lemma3.1 Sufficiency. Since $(\Omega,\mathcal{A},P)$ has a regular nest so it also has net $D^{*}=\{D_{i_1\cdots i_n}\}$ for $n\geq 1$, we use a transformation. (see [2])

$$\theta_1 : (i_1\cdots i_n) \longrightarrow (i_1-1,\cdots,i_n-1) ; \tag{3.7}$$

$$\theta_2 : (j_1\cdots j_n) \longrightarrow k \triangleq (j_1\cdot 2^{n-1}+j_2\cdot 2^{n-2}+\cdots+j_n)+1 \tag{3.8}$$

$\theta_2\circ\theta_1$ is a one-one correspondence between $\{(i_1\cdots i_n) \mid i_k=1,2;k=1,\ldots n\}$ and $\{1,2,\ldots,2^n\}$. Let $e=\theta_1^{-1}\circ\theta_2^{-1}$, indicate

$$B_k^{(n)} \triangleq \bigcup_{l=1}^{k} D_{e(k)}, \qquad (k=1,\ldots,2) \tag{3.9}$$

It is easy to see $B_k^{(n)}\in\mathcal{A}$, $P(B_k^{(n)})=k/2^n$, for any $\underset{\sim}{A}\in\mathcal{F}_1(U)$, i.e. $\underset{\sim}{A}$ is measurable, we define $G_{\underset{\sim}{A}}=\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^n} B_k^{(n)}\times\{u \mid \underset{\sim}{A}(u)\geq \frac{k}{2^n}\}$, and $\xi_{\underset{\sim}{A}}(\omega)=(G_{\underset{\sim}{A}})_\omega$. $\forall\omega\in\Omega$ , it is easy to test that $\xi_{\underset{\sim}{A}}$ is random set and $\mu_{\xi_{\underset{\sim}{A}}}=\underset{\sim}{A}$. We deduce $G_{\underset{\sim}{A}}\cup G_{\underset{\sim}{B}}= G_{\underset{\sim}{A}\cup\underset{\sim}{B}} \Longrightarrow \xi_{\underset{\sim}{A}\cup\underset{\sim}{B}}=\xi_{\underset{\sim}{A}}\cup\xi_{\underset{\sim}{B}} \Longrightarrow \mu_{\xi_{\underset{\sim}{A}}\cup\xi_{\underset{\sim}{B}}}=\underset{\sim}{A}\cup\underset{\sim}{B}$. We get the conclusion that $\{\xi_{\underset{\sim}{A}} \mid \underset{\sim}{A}\in\mathcal{F}_1(U)\}$ is delegation from the theorem2.2. $\square$

For a probability space owning a regular net, we may construct a complete falling measurable structure, and immediatly gain the inverse proposition of lemma3.2 from Theorem 3.1. By the way we get a result on probability space: A probability space has a regular nest iff it has a regular net.

<u>Theorem3.2</u> A complete falling measurable structure $(\Omega,\mathcal{A},P;U,\mathcal{B},\check{\mathcal{B}})$ has delegation iff $(\Omega,\mathcal{A},P)$ has regular net. $\square$

For next section's discussion, we introduce the following definition.

<u>Definition3.2</u> $\mathcal{D}=\{\xi_{\underset{\sim}{A}} \mid \underset{\sim}{A}\in\mathcal{F}_1(U), \mu_{\xi_{\underset{\sim}{A}}}=\underset{\sim}{A}\}$ is called single-nest delegation on the fall-shadow space $(\Omega,\mathcal{A},P;U,\mathcal{B},\check{\mathcal{B}})$ if there is a strict nest $\mathcal{T}$ on $(\Omega,\mathcal{A},P)$ such that $\forall\,\underset{\sim}{A}\in\mathcal{F}_1(U)$ and $\forall\, u\in U$ satisfys $\xi_{\underset{\sim}{A}}^{-1}(\dot{u})\in\mathcal{T}$ .

## §4. The measurability and the fallability

About the measurability and the fallability of fuzzy sets, [1] has got (see [1] Theorem2)

Theorem4.1 Let $(\Omega,\mathcal{A},P;U,\mathcal{B},\check{\mathcal{B}})$ be a fall-shadow space, if $(\Omega,\mathcal{A},P)$ is sufficient for$(\mathbb{R},\mathcal{B}_0)$,where$\mathcal{B}_0$ is the Borel field on$\mathbb{R}$,then a given fuzzy subset $\underset{\sim}{A}$ of U is always fallable provided that $\underset{\sim}{A}$ is $(\mathcal{B},\mathcal{B}_0)$-measurable.

We say that a probability space $(\Omega,\mathcal{A},P)$ is sufficient for a given measurable space $(X,\mathcal{B})$ if for any probability measure m defined on B,there is a$\mathcal{A}-\mathcal{B}$ measurable mapping $f:\Omega\to X$ such that

$$m(B)=P(\xi^{-1}(B)) \qquad \forall B\in\mathcal{B} \qquad (4.1)$$

[1] did not answer whether the invers prop. of Th4.1 is true. Here we weaken the need of the Th4.1 and in terms of delegation give a sufficient and necessary condition. It is pity that the problem is not completely sovled.

Lemma4.1 If $(\Omega,\mathcal{A},P)$ is sufficient for $(\mathbb{R},\mathcal{B}_0)$,the $(\Omega,\mathcal{A},P)$ has regular nest. []

Theorem4.2 Let $(\Omega,\mathcal{A},P;U,\mathcal{B},\check{\mathcal{B}})$ be a fall-shadow space,If $(\Omega,\mathcal{A},P)$ has regular nest,then any measurable fuzzy set A of U must be fallable.

Proof. Since we can construct a regular net from a regular nest,given a measurable fuzzy set $\underset{\sim}{A}$,we can make a random set $\xi_{\underset{\sim}{A}}$ such that$\mu_{\xi_{\underset{\sim}{A}}}=\underset{\sim}{A}$, by using the analogous method. []

Accoding to lemma4.1, we know the Th4.1(i.e.Th2 in 1 ) is the direct corollary of the Theorem.

Lemma4.2 Let $(\Omega,\mathcal{A},P;U,\mathcal{B},\check{\mathcal{B}})$ be a fall-shadow space,and $(\Omega,\mathcal{A},P)$ have a regular nest $\{B^{(\lambda)}\}_{\lambda\in[0,1]}$then fallable fuzzy set $\underset{\sim}{A}$ is measurable iff there exists a random set$\xi$ such that $\mu_\xi=\underset{\sim}{A}$ and $\{\xi^{-1}(\dot{u})\}_{u\in U}$is a strict nest.

Proof. Necessary. $\underset{\sim}{A}$ is measurable fuzzy set, $\forall u\in U$ let $F(u)=B^{(\lambda)}$ if $\underset{\sim}{A}(u)=\lambda$, thus this determines the mapping $F: U\to\mathcal{A}$.

Define $$\xi(\omega)=\{u\,|\,\omega\in F(u)\},\quad \forall\omega\in\Omega \qquad (4.2)$$

If $\xi$ is random set,then $\xi^{-1}(\dot{u})=F(u)$,therefore $\mu_\xi=\underset{\sim}{A}$ and $\{\xi^{-1}(u)\}$ become a strict nest.This shows need to only prove $\xi$ is random set,further more,we need only to show $\forall\omega\in\Omega$, $\xi(\omega)\in\mathcal{B}$. We have a natural correspondence between $\{B^{(\lambda)}\}_{\lambda\in[0,1]}$and $[0,1]$,so the topological properties of $[0,1]$ can be carried into$\{B^{(\lambda)}\}_{\lambda\in[0,1]}$, Thus we may suppose that $\{F(u_k)\}$ is the countable dense subset of $\{F(u)\}$ ,then

$E=\bigcap_{\omega\in F(u_n)} F(u_n) \in \mathcal{A}$ (4.3)

Let $P(E)=\lambda_0$, clearly to show

$$\{u \mid \underset{\sim}{A}(u)>\lambda_0\} \subset \xi(\omega) \subset \{u \mid \underset{\sim}{A}(u) \geqslant \lambda_0\} \tag{4.4}$$

If there is $u_0$, $\underset{\sim}{A}(u_0)=\lambda_0$ such that $u_0 \in \xi(\omega)$, then $\omega \in F(u_0)=B^{(\lambda_0)}$.

Besides, since $F(\{u \mid \underset{\sim}{A}(u)=\lambda_0)=B^{(\lambda_0)}=F(u_0)$, then $\forall u, \underset{\sim}{A}(u)=\lambda_0$, and $\omega \in F(u)$ i.e. $u \in \xi(\omega)$, so we have

$$\mathcal{F} \ni \xi(\omega)=\{u \mid \underset{\sim}{A}(u) \geqslant \lambda_0\} \text{ or } \{u \mid \underset{\sim}{A}(u) > \lambda_0\} \tag{4.5}$$

Sufficient. Suppose that there is $\xi$ shch that $\mathcal{M}_\xi=A$ and $\{\xi^{-1}(\dot{u})\}_{u\in U}$ is a strict nest, then $\xi(\omega)=\{u \mid \omega \in \xi^{-1}(\dot{u})\}$, For $\forall \lambda \in [0,1]$, if all u hold $\underset{\sim}{A}(u)>\lambda$ or all u hold $\underset{\sim}{A}(u)<\lambda$, then $\{u \mid \underset{\sim}{A}(u)>\lambda\}=\Omega$ or $\phi \in \mathcal{B}$. Now we assume there are some u hold $\underset{\sim}{A}(u)<\lambda$ and some u holds $A(u)>\lambda$, let $\mathcal{F}=\{\xi^{-1}(\dot{u}) \mid \underset{\sim}{A}(u)<\lambda\}$ be a strict nest, and $\{F^{(n)}\}_{n=1,2,\dots} \subset \mathcal{F}$ be increasing and $\forall \xi^{-1}(\dot{u}) \in \mathcal{F}$, there is some $N$: $F^{(n)} \supset \xi^{-1}(\dot{u})$ when $n \geq N$. Let $\mathcal{G}=\{\xi^{-1}(\dot{u}) \mid \xi^{-1}(\dot{u}) \supsetneqq \bigcup_{n=1}^{\infty} F^{(n)}\}$ then

$$\{u \mid \underset{\sim}{A}(u)>\lambda\}=\{u \mid \xi^{-1}(\dot{u}) \in \mathcal{G}\} \tag{4.6}$$

In fact, obviously $\{u \mid \underset{\sim}{A}(u)>\lambda\} \supset \{u \mid \xi^{-1}(\dot{u}) \in \mathcal{G}\}$, moreover $\forall u \in \{u \mid A(u)>\lambda\}$, when $u \notin \{u \mid \xi^{-1}(\dot{u}) \in \mathcal{G}\}$, imply $\xi^{-1}(\dot{u})=\bigcup_{n=1}^{\infty} F^{(n)}$, then $\underset{\sim}{A}(u)=P(\xi^{-1}(\dot{u}))=P(\bigcup_{n=1}^{\infty} F^{(n)}) \leq \lambda$, this is a contradictory.

Let $\{G^{(n)}\}_{n=1,2,\dots} \subset \mathcal{G}$ be a decreasing sequence and $\forall \xi^{-1}(\dot{u}) \in \mathcal{G}$, there is some $N$: $\xi^{-1}(\dot{u}) \supset G^{(n)}$ when $n \geq N$, If $\mathcal{G}$ has the minimal element $G$ then $G=\bigcap \mathcal{G}$, for any $\omega \in G$: $\xi(\omega)=\{u \mid \xi^{-1}(\dot{u}) \in \mathcal{G}\}=\{u \mid \underset{\sim}{A}(u)>\lambda\} \in \mathcal{B}$.

Now we suppose that $\mathcal{G}$ has no minimal element, without loss generality providing $\{G^{(n)}\}$ are different, drawing $\{\omega_n\}$ : $\omega_n \in G^{(n)}$, $\forall n$

$$\text{then } \{u \mid \xi^{-1}(\dot{u}) \in \mathcal{G}\} = \bigcup_{n=1}^{\infty} \xi(\omega_n) \tag{4.7}$$

In fact, always $\bigcup_{n=1}^{\infty} \xi(\omega_n) \subset \{u \mid \xi^{-1}(\dot{u}) \in \mathcal{G}\}$, on the other hand, drawing any $u \in \{u \mid \xi^{-1}(\dot{u}) \in \mathcal{G}\}$, there is some n: $\xi^{-1}(\dot{u}) \supset G^{(n)}$, then $\omega_n \in \xi^{-1}(\dot{u}) \Rightarrow u \in \xi(\omega_n) \subset \bigcup_{n=1}^{\infty} \xi(\omega_n)$, thus (4.7) is true, from (4.6) and (4.7) we know $\{u \mid \underset{\sim}{A}(u)>\lambda\} \in \mathcal{B}$. Consequently $\underset{\sim}{A}$ is measurable. []

<u>Theorem4.3</u> Let $(\Omega,\mathcal{A},P;U,\mathcal{B},\mathcal{\dot{B}})$ be fall-shadow space, and $(\Omega,\mathcal{A},P)$ have regular nest, then any fallable fuzzy set $\underset{\sim}{A}$ is measurable iff $\mathcal{F}(U)$ has single-nest delegation. []

On the codition given in [1], we have

Corollary4.1 Let $(\Omega,\mathcal{A},P;U,\mathcal{B},\check{\mathcal{B}})$ be a fall-shadow space, and $(\Omega,\mathcal{A},P)$ be sufficient for $(\mathbb{R},\mathcal{B}_0)$, then the fuzzy sets are measurable provided that they are fallable iff $\mathcal{F}_1(U)$ has single-nest delegation.

Reference

[1] Wang Pei-Zhuang, Sanchez, E. Treating a fuzzy set as a fallable random set, Fuzzy Information and Decision Process', Edited by M.M.Gupta, E.Sanchez, North-Holland Publishing Company, 1982, 213-219

[2] Wang Pei-Zhuang, Zhang Nian Lun, Fall-shadow space-probability description of fuzzy set, Journal of Math.Research and Exposition, vol.3 No.1 Jan(1983) 163-178

[3] Goodman, I.R. Fuzzysets as equivalence class of random sets, Recent Developments in Fuzzy Set and Possibility Theory, Edited by Yager, R.T. 1981

[4] Nguyen, H.T., ON random sets and belief functions, J.Math.Anal.&Applic., 65, 531-542, 1978

[5] Kwakernaak, H., Fuzzy random variables, I., INFO.SCI., 15, 1-29, 1978

[6] Kendall, D.G., Fundations of a theory of random sets, Stochastic Geormetry, New York, 322-376, 1974

[7] Zadeh.L.A., Fuzzy sets, Inform.and Control 8(1965), 338-353

[8] Zadeh.L.A., Fuzzy sets as a Basis for a Theory of Possibility, Memorandum M77/12, Electronic, Research Laboratory, Berkely, 1977