SQUARE LATTICE AND FUZZY NUMBER

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ABSTRACT

In this paper we try to establish the notion of a square lattice in order to lay an analytic foundation for those notions like fuzzy number and so on.

KEYWORDS

Square lattice; generalized distance; square body; fuzzy number; patially large fuzzy number; partially small fuzzy number.

1. SQUARE LATTICE

In all our reasoning here, X is always supposed to be a nonempty set, and x, y, z, u, v, w or a, b, c, etc. be members of X.

Definition 1.1. Let + be an algebraic operation on X, $\leqslant$ be a partial ordering on X. $\langle X, +, \leqslant\rangle$ is a square lattice iff

(Ⅰ) $\langle X, +\rangle$ is a commutative semigroup with zero element θ. If $x+y=x+z$, then $y=z$;

(Ⅱ) $\langle X, \leqslant\rangle$ is a lattice and order—complete(i.e. $\forall E\subseteq X$, if E has upper bounds, then E has a supremum).

(Ⅲ) $x\leqslant y$ iff $\exists z\in X, \theta\leqslant z$ such that $x+z=y$; $x+y=x\vee y$ iff $x\wedge y=\theta$, where $x\vee y$ and $x\wedge y$ denote respectively the supremum and the infimum of $\{x, y\}$.

"$x\leqslant y$" in our passage will be often denoted by "$y\geqslant x$".

Example 1. Let R^n be Euclid n—space, $x, y\in R^n$, $x=(x_1, x_2, \cdots, x_n)$, $y=(y_1, y_2, \cdots, y_n)$. Put $x+y=(x_1+y_1, x_2+y_2, \cdots, x_n+y_n)$; $x\leqslant y$ iff $x_i\leqslant y_i$ for $i=1, 2, \cdots, n$. Then $\langle R^n, +, \leqslant\rangle$ is a square lattice.

Example 2. Let N be the set of positive integers. when $m, n\in N$, put $m\hat{+}n=m+n-1$, $m\leqslant n$ iff $\frac{n}{m}\in N$. Then $\langle N, \hat{+}, \leqslant\rangle$ satisfies condition (Ⅰ) and (Ⅱ) in the definition 1.1. But it doesn't satisfies(Ⅲ), hence is not a square lattice.

The conditions in the definition 1.1 occur quite commonty. the mutual independence between them is obvious. Let's discuss some elementary properties of the square lattice.

Property 1. If $x\leqslant y$, then $x\wedge z\leqslant y\wedge z$, $x\vee z\leqslant y\vee z$ for arbitrary z.

Property 2. $x\wedge y\leqslant x\vee y$ for arbitrary x, y. $x\vee y=x\wedge y$ iff $x=y$.

Property 3. If $x\leqslant y$ and $u\leqslant v$, then $x+u\leqslant y+v$. In particular, if $x\leqslant y$, then $x+z\leqslant y+z$ for arbitrary z.

Proof. Since $x\leqslant y$, $\exists w_1\geqslant\theta$ such that $x+w_1=y$. Since $u\leqslant v$, $\exists w_2\geqslant\theta$ such that $u+w_2=v$. Thus $x+u+(w_1+w_2)=y+v$. $w_1+w_2\geqslant w_1\geqslant\theta$, and $x+u\leqslant y+v$ #

Property 4. If $x\leqslant y$, then there exists a unique $z\geqslant\theta$ such that $x+z=y$.

Proof. since $x\leqslant y$, by (Ⅲ) $\exists z\geqslant\theta$ such that $x+z=y$. In case $\exists z^*\geqslant\theta$ such that $x+z^*=y$.

Then $z^*=z$ by (I) #

For convenience' sake, we'll introduce some more signs.

Definition 1.2. Let X be a square lattice. $x, y, z \in X$. We write $z=y-x$ iff $x+z=y$.

By (I) and property 4 it is easily seen that $y-x$ is uniquely determined by x and y. However $\langle X, + \rangle$ is only a semigroup, $-x$ is not necessarily a member of X, hence $y-x$ is an integral sign. From (II) we deduce that when $x \leqslant y$, $y-x \in X$, at the same time $y-x \geqslant \theta$. From the definition we get directly that if $y-x \in X$, then $(y-x)+x=y$. what's more, $x-x=\theta$. And, if $x+y \leqslant z$, then $x \leqslant z-y$.

Property 5. Let $x, x_\lambda \in X$, $\forall \lambda \in \Lambda$, and $\{x_\lambda\}_{\lambda\in\Lambda}$ be bounded. Then

$$x+\bigvee_{\lambda\in\Lambda} x_\lambda=\bigvee_{\lambda\in\Lambda}(x+x_\lambda),$$

$$x+\bigvee_{\lambda\in\Lambda} x_\lambda=\bigwedge_{\lambda\in\Lambda}(x+x_\lambda).$$

Proof. Since $x_\lambda \leqslant \bigwedge_{\lambda\in\Lambda} x_\lambda$, $\forall \lambda \in \Lambda$, so that $x+x_\lambda \leqslant x+\bigvee_{\lambda\in\Lambda} x_\lambda$ for all $\lambda \in \Lambda$. Hence

$$\bigvee_{\lambda\in\Lambda}(x+x_\lambda) \leqslant x+\bigvee_{\lambda\in\Lambda} x_\lambda.$$

Also, since $x+x_\lambda \leqslant \bigvee_{\lambda\in\Lambda}(x+x_\lambda)$, $\forall \lambda \in \Lambda$, then $x_\lambda \leqslant \bigvee_{\lambda\in\Lambda}(x+x_\lambda)-x$, $\forall \lambda \in \Lambda$, so that $\bigvee_{\lambda\in\Lambda} x_\lambda \leqslant \bigvee_{\lambda\in\Lambda}(x+x_\lambda)-x$. Hence

$$x+\bigvee_{\lambda\in\Lambda} x_\lambda \leqslant \bigvee_{\lambda\in\Lambda}(x+x_\lambda).$$

Thus

$$x+\bigwedge_{\lambda\in\Lambda} x_\lambda=\bigwedge_{\lambda\in\Lambda}(x+x_\lambda).$$

The proof of the second assertion is analogous #

In particular, for arbitrary $x, y, z \in X$, we have

$$x+y\vee z=(x+y)\vee(x+z), \quad x+y\wedge z=(x+y)\wedge(x+z).$$

Property 6. $x+y=x\wedge y+x\vee y$

Proof. Let $x\wedge y+u=x$, $x\wedge y+v=y$. By property 5,

$$x\wedge y+u\wedge v=(x\wedge y+u)\wedge(x\wedge y+v)=x\wedge y.$$

By property 4, we obtain $u\wedge v=\theta$. By(II), $u\vee v=u+v$. So that

$$x\wedge y+u+v=x\wedge y+u\vee v=(x\wedge y+u)\vee(x\wedge y+v)=x\vee y.$$

Hence

$$x\wedge y+u+x\wedge y+v=x\vee y+x\wedge y.$$

That is

$$x+y=x\vee y+x\wedge y \ \#$$

Theorem 1.1 . Let X be a square lattice. Then, for arbitrary $x, y \ z \in X$ we have

$$x\vee y-x\wedge y \leqslant (x\vee z-x\wedge z)+(z\vee y-z\wedge y).$$

Proof. By property 6, we have

$$x\wedge z+z\wedge y=(x\wedge z)\vee(z\wedge y)+(x\wedge z)\wedge(z\wedge y) \leqslant z\wedge(x\vee y)+z\wedge(x\wedge y).$$

Since $z\wedge(x\vee y) \leqslant z$, $z\wedge(x\wedge y) \leqslant x\wedge y$. By property 3, we obtain $x\wedge z+z\wedge y \leqslant z+x\wedge y$, So that

$$x+y+x\wedge z+z\wedge y+x\wedge z+z\wedge y \leqslant x+z+y+z+x\wedge y+x\wedge y.$$

With property 6, we get immediately the result.

2. CONVERGENCE PROPERTIES OF SEQUENCES IN A SQUARE LATTICE

Definition 2.1. Let X be a square lattice. $x_n \in X$. $\{x_n\}_{n\in N}$ be bounded. Write

$$\overline{\lim_{n\to\infty}} x_n=\bigwedge_{n=1}^{\infty}\bigvee_{k=n}^{\infty} x_k, \qquad \underline{\lim_{n\to\infty}} x_n=\bigvee_{n=1}^{\infty}\bigwedge_{k=n}^{\infty} x_k.$$

$\overline{\lim}_{n\to\infty} x_n$ and $\underline{\lim}_{n\to\infty} x_n$ is called superior limit and inferior limit of $\{x_n\}$, respectively. If, and only if $\underline{\lim}_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n = x$, we say that the limit of $\{x_n\}$ exists, and we write $x_n \to x$, or $\lim_{n\to\infty} x_n = x$.

Obviously, if $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant \cdots$, and is lower bounded, then $\lim_{n\to\infty} x_n$ exists, and

$$x = \lim_{n\to\infty} x_n = \bigwedge_{n=1}^{\infty} x_n$$

If $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \cdots$, and is upper bounded, then $\lim_{n\to\infty} x_n$ exists, and

$$x = \lim_{n\to\infty} x_n = \bigvee_{n=1}^{\infty} x_n$$

In addition, $\underline{\lim}_{n\to\infty} x_n \leqslant \overline{\lim}_{n\to\infty} x_n$ is obvious.

Theorem 2.1. Let X be a square lattice, $x_n, y_n \in X$, $\forall n \in N$. $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ exist, then

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n .$$

Proof. It is easily proved that

$$\bigvee_{n=1}^{\infty} (x_n + y_n) \leqslant \bigvee_{n=1}^{\infty} x_n + \bigvee_{n=1}^{\infty} y_n.$$

for arbitrary $\{x_n\}$ and $\{y_n\}$.

Now, suppose that $x_n \uparrow$, $y_n \uparrow$, and $x = \bigvee_{n=1}^{\infty} x_n$, $y = \bigvee_{n=1}^{\infty} y_n$, $u = \bigvee_{n=1}^{\infty} (x_n + y_n)$. Since x_n and y_n are monotone increasing, $x_n + y_m \leqslant u$ for arbitrary n and m. Keeping m fixed, and taking the sup for n, from property 5 in section 1 we obtain $x + y_m \leqslant u$, $\forall m$. Also, taking the sup for m, we get $x + y \leqslant u$. Thus, when $x_n \uparrow x$ and $y_n \uparrow y$, the assertion is true. The same assertion holds when applied to the case where $x_n \downarrow x$ and $y_n \downarrow y$.

Since $\bigvee_{k=n}^{\infty} (x_k + y_k) \leqslant \bigvee_{k=n}^{\infty} x_k + \bigvee_{k=n}^{\infty} y_k$, $\forall n$, but $\bigvee_{k=n}^{\infty} x_k \downarrow$, $\bigvee_{k=n}^{\infty} y_k \downarrow$, so we obtain

$$\overline{\lim}_{n\to\infty} (x_n + y_n) \leqslant \overline{\lim}_{n\to\infty} x_n + \overline{\lim}_{n\to\infty} y_n = x + y.$$

On the other hand, since

$$\bigwedge_{k=n}^{\infty} x_k + \bigwedge_{k=n}^{\infty} y_k \leqslant \bigwedge_{k=n}^{\infty} (x_k + y_k), \quad \forall n$$

but $\bigwedge_{k=n}^{\infty} x_n \uparrow$, $\bigwedge_{k=n}^{\infty} y_n \uparrow$, we obtain

$$\lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = \underline{\lim}_{n\to\infty} x_n + \underline{\lim}_{n\to\infty} y_n \leqslant \underline{\lim}_{n\to\infty} (x_n + y_n).$$

Hence

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n \ *$$

Theorem 2.2. Let X be a square lattice. $x_n \in X$, $\forall n$. Then $x_n \to x$ iff $x \wedge x_n \to x$ and $x \vee x_n \to x$.

Proof. Suppose that $x_n \to x$, we have

$$x = \bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} x_k \leqslant \bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} (x \vee x_k) \leqslant \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} (x \vee x_k) =$$

$$\bigwedge_{n=1}^{\infty} (x \vee (\bigvee_{k=n}^{\infty} x_k)) = \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} x_k = x$$

So, $x \vee x_n \to x$. $x \wedge x_n \to x$ can also be proved in the same way.

Suppose that $x \vee x_n \to x$ and $x \wedge x_n \to x$. Since

$$x \wedge x_n \leqslant x_n \leqslant x \vee x_n$$

Hence $x_n \to x$ #

Definition 2.2. Let X be a square lattice. $x, y \in X$. Put

$$\rho(x, y) = x \vee y - x \wedge y,$$

$\rho(x, y)$ is called a generalized distance of x and y.

Through definition 2.2 and theorem 1.1 in section 1, we clearly have:

(1) $\rho(x, y) \geqslant \theta$; $\rho(x, y) = \theta$ iff $x = y$.

(2) $\rho(x, y) = \rho(y, x)$, $\forall x, y \in X$.

(3) $\rho(x, y) \leqslant \rho(x, z) + \rho(z, y)$, $\forall x, y, z \in X$.

Theorem 2.3. Let X be a square lattice. $x_n \in X$. $x_n \to x$ iff

$$\lim_{n\to\infty} \rho(x, x_n) = \theta.$$

Proof. If $x_n \to x$, then $x \vee x_n \to x$ and $x \wedge x_n \to x$ by theorem 2.2. Since

$$\bigvee_{k=n}^{\infty} (x \vee x_k) = \left(\bigvee_{k=n}^{\infty} (x \vee x_k) - \bigwedge_{k=n}^{\infty} (x \wedge x_k) \right) + \bigwedge_{k=n}^{\infty} (x \wedge x_k).$$

but $\left(\bigvee_{k=n}^{\infty} (x \vee x_k) - \bigwedge_{k=n}^{\infty} (x \wedge x_k)\right) \downarrow$, and it is of lower bound θ, letting $n \to \infty$, by **theorem 2.1** we obtain

$$x = \lim_{n\to\infty} \left(\bigvee_{k=n}^{\infty} (x \vee x_k) - \bigwedge_{k=n}^{\infty} (x \wedge x_k) \right) + x.$$

Hence

$$\lim_{n\to\infty} \left(\bigvee_{k=n}^{\infty} (x \vee x_k) - \bigwedge_{k=n}^{\infty} (x \wedge x_k) \right) = \theta.$$

Notice that

$$\theta \leqslant x \vee x_n - x \wedge x_n \leqslant \bigvee_{k=n}^{\infty} (x \vee x_k) - \bigwedge_{k=n}^{\infty} (x \wedge x_k).$$

Thus it can immediately be obtained

$$\lim_{n\to\infty} (x \vee x_n - x \wedge x_n) = \theta,$$

i.e. $\lim_{n\to} \rho(x, x_n) = \theta$

Conversely, let $x \vee x_n - x \wedge x_n \to \theta$. Since

$$x \vee x_n = (x \vee x_n - x \wedge x_n) + x \wedge x_n.$$

So that

$$x \leqslant x \vee x_n \leqslant \bigvee_{k=n}^{\infty} (x \vee x_k - x \wedge x_k) + x, \quad \forall n.$$

Thereby

$$\overline{\lim_{n\to\infty}} (x \vee x_n) = \underline{\lim_{n\to\infty}} (x \vee x_n) = x.$$

That is

$$x \vee x_n \to x.$$

By $x \vee x_k \leqslant \bigvee_{k=n}^{\infty} (x \vee x_k - x \wedge x_k) + x \wedge x_k$, $\forall k \geqslant n$, with property 5 in section 1, we obtain

$$\bigwedge_{k=n}^{\infty} (x \vee x_k) \leqslant \bigvee_{k=n}^{\infty} (x \vee x_k - x \wedge x_k) + \bigwedge_{k=n}^{\infty} (x \wedge x_k).$$

Hence

$$x = \underline{\lim_{n\to\infty}} (x \vee x_n) \leqslant \underline{\lim_{n\to\infty}} (x \wedge x_n) \leqslant \overline{\lim_{n\to\infty}} (x \wedge x_n) \leqslant \overline{\lim_{n\to\infty}} (x \vee x_n) = x.$$

That is $x_n \wedge x \to x$. Thereby $x_n \to x$. #

Theorem 2.4. Let X be a square lattice. $x_n \in X$, $\forall n$. $\lim_{n\to\infty} x_n$ exists. Then it is necessarily unique.

Proof. Let $x_n \to x$, also, $x_n \to y$. Then

$$\rho(x,y)\leqslant\rho(x,x_n)+\rho(x_n,y)\to\theta,\ \text{as}\ n\to\infty,$$

i.e. $\rho(x,y)=\theta$. Thereby $x=y$. #

Definition 2.3. Let X be a square lattice. $x_n\in X$, $\forall n$. If there exist $u_n\in X$, $u_1\geqslant u_2\geqslant\cdots\geqslant u_n\geqslant\cdots$, $\bigwedge_{n=1}^{\infty} u_n=\theta$ such that

$$\rho(x_n,x_m)\leqslant u_N,\ \text{as}\ n,\ m\geqslant N,$$

then $\{x_n\}$ is called a generalized elementary sequence.

Theorem 2.5. The generalized elementary sequence in a square lattice is convergent.

Proof. Since $x_1\vee x_n-x_1\wedge x_n\leqslant u_1$, $\forall n$. Hence

$$x_1\vee x_n\leqslant u_1+x_1\wedge x_n\leqslant u_1+x_1,\ \forall n.$$

Thereby $\bigvee_{n=1}^{\infty} x_n\leqslant u_1+x_1$, i.e. $\{x_n\}$ is upper bounded. Similarly, $\{x_n\}$ is lower bounded. Put

$$\overline{\lim_{n\to\infty}}\, x_n=x,\quad \underline{\lim_{n\to\infty}}\, x_n=y,$$

then

$$\rho(x,\ \bigvee_{k=n}^{\infty}x_k),\ \rho(\bigwedge_{k=n}^{\infty}x_k,\ y)\to\theta,\ \text{as}\ n\to\infty.$$

But

$$\theta\leqslant\rho(x,\ y)\leqslant\rho(x,\ \bigvee_{k=n}^{\infty}x_k)+\rho(\bigvee_{k=n}^{\infty}x_k,\ \bigwedge_{k=n}^{\infty}x_k)+\rho(\bigwedge_{k=n}^{\infty}x_k,\ y),$$

Since $x_n\vee x_k-x_n\wedge x_k\leqslant u_n$, as $k\geqslant n$, i.e. $x_n\vee x_k\leqslant u_n+x_n\wedge x_k$, by $x_n\wedge x_k\leqslant x_n$ we see that $x_n\vee x_k\leqslant u_n+x_n$, $\forall k\geqslant n$. Hence $\bigvee_{k=n}^{\infty}x_k\leqslant u_n+x_n$. By $x_n\vee x_k\geqslant x_n$ we see that $x_n\leqslant u_n+x_n\wedge x_k$, $\forall k\geqslant n$. Hence $x_n\leqslant u_n+\bigwedge_{k=n}^{\infty}x_k$. Thereby

$$\rho(\bigvee_{k=n}^{\infty}x_k,\ \bigwedge_{k=n}^{\infty}x_k)=\bigvee_{k=n}^{\infty}x_k-\bigwedge_{k=n}^{\infty}x_k\leqslant u_n+u_n\to\theta,\ \text{as}\ n\to\infty.$$

Hence $\rho(x,y)=\theta$, i.e. $x=y$. #

Definition 2.4. Let X be a square lattice. a, $b\in X$ and $a\leqslant b$. Put

$$[a,b]=\{x\,|\,a\leqslant x\leqslant b\},$$

$[a,b]$ is called a square body on X.

Theorem 2.6. Let $[a_n,b_n]$ be a square body on X for all $n\in N$, and $[a_1,b_1]\supseteq[a_2,b_2]\supseteq\cdots\supseteq[a_n,b_n]\supseteq\cdots$, $\rho(a_n,b_n)\to\theta$, as $n\to\infty$. Then there exists a unique $x\in X$ such that $x\in[a_n,b_n]$ for all $n\in N$.

Proof. Put $x=\bigvee_{n=1}^{\infty}a_n$, $y=\bigwedge_{n=1}^{\infty}b_n$, then for all $n\in N$, we have $x,y\in[a_n,b_n]$. Since

$$\theta\leqslant\rho(x,y)\leqslant\rho(x,a_n)+\rho(a_n,b_n)+\rho(b_n,\ y)\to\theta,$$

so that $x=y$. #

Definition 2.5. Let X and Y be two square lattices. $f: X\to Y$ is continuous at $x\in X$ iff for arbitrary $x_n\to x$ implies $f(x_n)\to f(x)$. If f is continuous at every point of X, then f is said to be a continuous mapping of X into Y.

Theorem 2.7. Suppose f and g are mappings of X into Y and they are continuous at x. Put

$$(f+g)(x)=f(x)+g(x),$$

then $f+g$ is also continuous at x.

Topological structure of a square lattice given by a square body will be dealt with in another paper.

3. FUZZY NUMBER

Definition 3.1. Suppose that $\underset{\sim}{A}$ is a fuzzy subset on R, $\mu_{\underset{\sim}{A}}$ is its membership function, and suppose it satisfies the following conditions:

(1) For arbitrary $\alpha \in (0, 1]$, $\underset{\sim}{A}_{\bar{\alpha}} = \{x \mid \mu_{\underset{\sim}{A}}(x) \geqslant \alpha\}$ is a bounded closed interval;

(2) There exists a unique $a \in R$ such that $\mu_{\underset{\sim}{A}}(a) = 1$.

Then $\underset{\sim}{A}$ is called a fuzzy number or fuzzy a, which we denote by $\underset{\sim}{a}$. For convenience' sake, we use the $\underset{\sim}{a}(x)$ to denote the grade of membership of x in $\underset{\sim}{a}$.

By definition 3.1 it is easily seen that $\underset{\sim}{a}$ is a convex fuzzy subset of R, i.e. $\underset{\sim}{a}(x) \wedge \underset{\sim}{a}(z) \leqslant \underset{\sim}{a}(y)$ for arbitrary $x \leqslant y \leqslant z$. A general real number a can be regarded as a fuzzy number, and

$$\underset{\sim}{a}(x) = \begin{cases} 1, & \text{iff } x = a; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, suppose that $\underset{\sim}{a}$ is a fuzzy number and $\underset{\sim}{a}(x) = 0$ for arbitrary $x < a$, then $\underset{\sim}{a}$ is called a partially large fuzzy number. If $\underset{\sim}{a}(x) = 0$ for arbitrary $x > a$, then $\underset{\sim}{a}$ is called a partially small fuzzy number. In the following passage we shall use FN(R) for all fuzzy numbers, FM(R) for all partially large fuzzy numbers and FS(R) for all partially small fuzzy numbers.

If $\underset{\sim}{a} \in FN(R)$, write $\underset{\sim}{a}_{\bar{\alpha}} = [m_\alpha^{(a)}, M_\alpha^{(a)}]$, then $\underset{\sim}{a}_{\bar{\alpha}} = [a, M_\alpha^{(a)}]$ when $\underset{\sim}{a} \in FM(R)$; $\underset{\sim}{a}_{\bar{\alpha}} = [m_\alpha^{(a)}, a]$ when $\underset{\sim}{a} \in FS(R)$.

Definition 3.2. Suppose $\underset{\sim}{a}$, $\underset{\sim}{b} \in FM(R)$,

(1) $\underset{\sim}{a} + \underset{\sim}{b} \overset{\Delta}{=} \bigcup_{\alpha \in (0,1]} \alpha \cdot [a+b, M_\alpha^{(a)} + M_\alpha^{(b)}]$.

(2) $\underset{\sim}{a} \leqslant^* \underset{\sim}{b}$ iff $0 \leqslant M_\beta^{(b)} - M_\beta^{(a)} \leqslant M_\alpha^{(b)} - M_\alpha^{(a)}$, for arbitrary $\alpha \leqslant \beta$ $(\alpha, \beta \in (0, 1])$.

Lemma 1. $\langle FM(R), + \rangle$ is a commutative semigroup and (i) general number 0 is its zero element; (ii) if $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{a} + \underset{\sim}{c}$, then $\underset{\sim}{b} = \underset{\sim}{c}$.

Lemma 2. Let $\underset{\sim}{a}$, $\underset{\sim}{b} \in FM(R)$, $\underset{\sim}{a} \leqslant^* \underset{\sim}{b}$ iff $\exists\, \underset{\sim}{u} \in FM(R)$ $(\theta \leqslant^* \underset{\sim}{u})$ such that $\underset{\sim}{a} + \underset{\sim}{u} = \underset{\sim}{b}$.

Proof Suppose that $\underset{\sim}{a} \leqslant^* \underset{\sim}{b}$, then $a \leqslant b$. Put

$$\underset{\sim}{u} = \bigcup_{\alpha \in (0,1]} \alpha \cdot [b-a, M_\alpha^{(b)} - M_\alpha^{(a)}].$$

It is easily seen that $\underset{\sim}{u} \in FM(R)$ and $\underset{\sim}{a} + \underset{\sim}{u} = \underset{\sim}{b}$.

Conversely, suppose that $\theta \leqslant^* \underset{\sim}{u}$ and $\underset{\sim}{a} + \underset{\sim}{u} = \underset{\sim}{b}$. Since $u \geqslant 0$ and $u \leqslant M_\beta^{(u)} \leqslant M_\alpha^{(u)}$ for arbitrary $\alpha \leqslant \beta$. Hence

$$0 \leqslant M_\beta^{(u)} = M_\beta^{(b)} - M_\beta^{(a)} \leqslant M_\alpha^{(u)} = M_\alpha^{(b)} - M_\alpha^{(a)}. *$$

Lemma 3. $\langle FM(R), \leqslant^* \rangle$ is a lattice and (i) $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{a} \vee \underset{\sim}{b} + \underset{\sim}{a} \wedge \underset{\sim}{b}$; (ii) $\leqslant^*$ is order—complete.

Proof Let $\underset{\sim}{a}$, $\underset{\sim}{b} \in FM(R)$ be given and $a \leqslant b$. Put

$$\underset{\sim}{u}_0 = \bigcup_{\alpha \in (0,1]} \alpha \cdot [b-a, M_\alpha^{(b)} - a], \quad \underset{\sim}{v}_0 = \bigcup_{\alpha \in (0,1]} \alpha \cdot [0, M_\alpha^{(a)} - a].$$

Then $\underset{\sim}{u}_0$, $\underset{\sim}{v}_0 \in FM(R)$ and $\underset{\sim}{a} + \underset{\sim}{u}_0 = \underset{\sim}{b} + \underset{\sim}{v}_0 \overset{\Delta}{=} \underset{\sim}{b}^*$.

Now suppose that $\underset{\sim}{u}_\lambda, \underset{\sim}{v}_\lambda \in FM(R)$, $\forall \lambda \in \Lambda$, and $\{\underset{\sim}{u}_\lambda\}_{\lambda \in \Lambda}$, $\{\underset{\sim}{v}_\lambda\}_{\lambda \in \Lambda}$ are sets of all $\underset{\sim}{u}$, $\underset{\sim}{v}$, respectively which satisfy $\underset{\sim}{a} + \underset{\sim}{u} = \underset{\sim}{b} + \underset{\sim}{v} \leqslant \underset{\sim}{b}^*$. We have

$$(\underset{\sim}{u}_\lambda)_{\bar{\alpha}} = [b-a, M_\alpha^{(u_\lambda)}], \; (\underset{\sim}{v}_\lambda)_{\bar{\alpha}} = [0, M_\alpha^{(v_\lambda)}], \; \forall \lambda \in \Lambda$$

Obviously $\Lambda \neq \phi$. Put

$$M_{\cdot}^{(u)}=\inf_{\lambda\in\Lambda} M_{\alpha}^{(u_\lambda)}, \qquad M_{\lambda}^{(v)}=\inf_{\lambda\in\Lambda} M_{\alpha}^{(v_\lambda)}.$$

Write

$$\underset{\sim}{u}=\bigcup_{\alpha\in(0,1]}\alpha\cdot[b-a,\ M_{\alpha}^{(u)}], \qquad \underset{\sim}{v}=\bigcup_{\alpha\in(0,1]}\alpha\cdot[0,M_{\alpha}^{(V)}].$$

Since if $\alpha\leqslant\beta$, then $b-a\leqslant M_{\beta}^{(u_\lambda)}\leqslant M_{\alpha}^{(u_\lambda)}$, $\forall\lambda\in\Lambda$. Thereby $b-a\leqslant M_{\beta}^{(u)}\leqslant M_{\alpha}^{(u)}$ and $M_1^{(u)}=b-a$, therefore $\underset{\sim}{u}\in FM(R)$. The proof of $\underset{\sim}{v}\in FM(R)$ is done in the same way.

We always have

$$b\leqslant M_{\alpha}^{(a)}+M_{\alpha}^{(u_\lambda)}=M_{\alpha}^{(b)}+M_{\alpha}^{(v_\lambda)}\leqslant M_{\alpha}^{(a)}+M_{\alpha}^{(b)}$$

for $\forall\lambda\in\Lambda$ and $\forall\alpha\in(0,1]$. Hence

$$b\leqslant M_{\alpha}^{(a)}+\inf_{\lambda\in\Lambda}M_{\alpha}^{(u_\lambda)}=M_{\alpha}^{(b)}+\inf_{\lambda\in\Lambda}M_{\alpha}^{(v_\lambda)}\leqslant M_{\alpha}^{(a)}+M_{\alpha}^{(b)}.$$

Thereby $\underset{\sim}{a}+\underset{\sim}{u}=\underset{\sim}{b}+\underset{\sim}{v}\leqslant\underset{\sim}{b}^*$. Put

$$\underset{\sim}{b}^{**}=\underset{\sim}{a}+\underset{\sim}{u}=\underset{\sim}{b}+\underset{\sim}{v},$$

then $\underset{\sim}{b}^{**}$ is an upper bound of $\{\underset{\sim}{a},\ \underset{\sim}{b}\}$, it is also a supremum of $\{a,\ b\}$.

Put $\underset{\sim}{a}^*=(\underset{\sim}{a}+\underset{\sim}{b})-\underset{\sim}{b}^{**}$, It is easily proved that $\underset{\sim}{a}^*$ is an infimum of $\{a,b\}$. Hence $\langle FM(R),\leqslant^*\rangle$ is a lattice and we have $\underset{\sim}{a}^*+\underset{\sim}{b}^{**}=\underset{\sim}{a}\wedge\underset{\sim}{b}+\underset{\sim}{a}\vee\underset{\sim}{b}=\underset{\sim}{a}+\underset{\sim}{b}$ evidently.

Now suppose that $\underset{\sim}{a}_\lambda\in FM(R)$, $\forall\lambda\in\Lambda$, $\{\underset{\sim}{a}_\lambda\}_{\lambda\in\Lambda}$ are bounded above, and $A\sqsubseteq FM(R)$ is a set of all upper bound of $\{\underset{\sim}{a}_\lambda\}_{\lambda\in\Lambda}$, then $A\neq\phi$. Let $\underset{\sim}{u}\in A$. Put $\underset{\sim}{u}_{\alpha}=[u,M_{\alpha}^{(u)}]$, and write

$$a=\inf_{u\in A} u, \qquad M_{\alpha}^{(a)}=\inf_{\underset{\sim}{u}\in A} M_{\alpha}^{(u)}$$

We can also prove that $\underset{\sim}{a}=\bigcup_{\alpha\in(0,1]}\alpha\cdot[a,M_{\alpha}^{(a)}]$ is a supremum of $\{\underset{\sim}{a}_\lambda\}_{\lambda\in\Lambda}$. Thereby $\leqslant^*$ is order—Complete. #

Theorem 3.1. $\langle FM(R),+,\leqslant^*\rangle$ is a square lattice.

Notice that $\underset{\sim}{a}\in FS(R)$, then $\underset{\sim}{a}_{\alpha}=[m_{\alpha}^{(a)},a]$. Put

$$-\underset{\sim}{a}=\bigcup_{\alpha\in(0,1]}\alpha\cdot[-a,\ -m_{\alpha}^{(a)}],$$

then $-\underset{\sim}{a}\in FM(R)$.

Definition 3.3. Let $\underset{\sim}{a},\ \underset{\sim}{b}\in FS(R)$.

(1) $\underset{\sim}{a}+\underset{\sim}{b}\overset{\triangle}{=}-[(-\underset{\sim}{a})+(-\underset{\sim}{b})]$.

(2) $\underset{\sim}{a}\leqslant^{**}\underset{\sim}{b}$ iff $-\underset{\sim}{a}\leqslant^*-\underset{\sim}{b}$ in FM(R).

Theorem 3.2. $\langle FS(R),+,\leqslant^{**}\rangle$ is a square lattice.

Observe that if $\underset{\sim}{a}\in FN(R)$, then $\underset{\sim}{a}_{\alpha}=[m_{\alpha}^{(a)},\ M_{\alpha}^{(a)}]$.

Let

$$\underset{\sim}{a}^{**}=\bigcup_{\alpha\in(0,1]}\alpha\cdot[m_{\alpha}^{(a)},a], \quad \underset{\sim}{a}^*=\bigcup_{\alpha\in(0,1]}\alpha\cdot[a,M_{\alpha}^{(a)}],$$

then $\underset{\sim}{a}^{**}\in FS(R)$, $\underset{\sim}{a}^*\in FM(R)$. We write $\underset{\sim}{a}=\underset{\sim}{a}^{**}\hat{+}\underset{\sim}{a}^*$.

Definition 3.4. Let $\underset{\sim}{a},\ \underset{\sim}{b}\in FN(R)$,

(1) $\underset{\sim}{a}+\underset{\sim}{b}\overset{\triangle}{=}(\underset{\sim}{a}^*+\underset{\sim}{b}^*)\hat{+}(\underset{\sim}{a}^{**}+\underset{\sim}{b}^{**})$.

(2) $\underset{\sim}{a}\leqslant\underset{\sim}{b}$ iff $\underset{\sim}{a}^*\leqslant^*\underset{\sim}{b}^*$ in FM(R) and $\underset{\sim}{a}^{**}\leqslant^{**}\underset{\sim}{b}^{**}$ in FS(R).

Theorem 3.3. $\langle FN(R),+,\leqslant\rangle$ is a square lattice.

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