Extension Principle and its Use in Fuzzy Optimization

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1. INTRODUCTION

The Extension principle introduced by L.A.Zadeh plays most important role in fuzzy set theory, see e.g. [1] . In the present paper we shall deal with two possibilities of generalization of this principle to set-to-set mappings. We present several relations between the two possibilities one of which turns out to be suitable for introducing the notion of optimal solution of a fuzzy optimization problem.

2. PRELIMINARIES

In the paper [1] , the Extension principle reads as follows: Let X , Y be sets , G be a point-to-point mapping from X to Y , i.e. $G : X \longrightarrow Y$, $\underset{\sim}{A}$ be a fuzzy set on X , i.e. $\underset{\sim}{A} \subsetneqq X$. The image of the of the fuzzy set $\underset{\sim}{A}$ by G is a fuzzy set $G(\underset{\sim}{A}) \subsetneqq Y$ such that

$$G(\underset{\sim}{A})y = 0 \vee \bigvee_{\substack{y=G(x) \\ x \in X}} \underset{\sim}{A}x , \tag{1}$$

where $\underset{\sim}{A}x$ means the value of the membership function of $\underset{\sim}{A}$, i.e. $\underset{\sim}{A}x \in [0,1]$ and $[0,1]$ is a unit interval in the real line R. By the symbol $\bigvee$ we mean supremum (maximum) , the set of all subsets of X is denoted by $\mathfrak{P}(X)$, the set of all fuzzy sets on X

is denoted by $\underset{\sim}{\mathcal{P}}(X)$.

3. EXTENSION PRINCIPLE

We shall introduce two definitions of images of a fuzzy set in case of set-to-set mappings.

<u>Definition 1 .</u> Let G be a set-to-set mapping, i.e. $G : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$, $\underset{\sim}{A}$ being a fuzzy set on X , i.e. $\underset{\sim}{A} \underset{\sim}{\subseteq} X$. The image of the fuzzy set $\underset{\sim}{A}$ by G be a fuzzy set $G_1(\underset{\sim}{A}) \underset{\sim}{\subseteq} Y$ defined by the following formula:

$$(2) \qquad G_1(\underset{\sim}{A})y = 0 \vee \bigvee_{\substack{y \in G(\{x\}) \\ x \in X}} \underset{\sim}{A}x$$

for all $y \in Y$.

<u>Definition 2 .</u> Let $G : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$, $\underset{\sim}{A} \underset{\sim}{\subseteq} X$. The image of the fuzzy set $\underset{\sim}{A}$ by G be a fuzzy set $G_2(\underset{\sim}{A}) \underset{\sim}{\subseteq} Y$ defined by the following formula:

$$(3) \qquad G_2(\underset{\sim}{A})y = 0 \vee \bigvee_{\substack{y \in G(\underset{\sim}{A}_\alpha) \\ \alpha \in [0,1]}} \alpha$$

for all $y \in Y$, $\underset{\sim}{A}_\alpha = \{x \in X \ ; \ \underset{\sim}{A}x \geqq \alpha\}$ being the α-level set of $\underset{\sim}{A}$.

Set

$$(4) \qquad \mathcal{V}_1(X,Y) = \{G \ ; \ G: \mathcal{P}(X) \rightarrow \mathcal{P}(Y),\ U \subset V \subset X \Rightarrow G(U) \subset G(V)\} ,$$

$$(5) \qquad \mathcal{V}_2(X,Y) = \{G \ ; \ G: \mathcal{P}(X) \rightarrow \mathcal{P}(Y),\ U \subset X \Rightarrow G(U) \subset \bigcup_{x \in U} G(\{x\})\} .$$

<u>Assertion 1 .</u> If $G \in \mathcal{V}_1$, $\underset{\sim}{A} \underset{\sim}{\subseteq} X$, then

$$(6) \qquad G_1(\underset{\sim}{A}) \underset{\sim}{\subseteq} G_2(\underset{\sim}{A}) ,$$

if $G \in \mathcal{V}_2$, $\underset{\sim}{A} \underset{\sim}{\subseteq} X$, then

(7) $$G_1(\underset{\sim}{A}) \underset{\sim}{\supseteq} G_2(\underset{\sim}{A}) \;.$$

<u>Corollary</u>. Let $G : X \longrightarrow Y$ be a point-to-point mapping. This mapping may be understood as a set-to-set mapping by a natural way setting

$$G(U) = \bigcup_{u \in U} \{G(u)\} \;.$$

It is easy to see that $G \in \vartheta_1 \cap \vartheta_2$, consequently, by Assertion 1 for $\underset{\sim}{A} \subseteq X$ we have

(8) $$G_1(\underset{\sim}{A}) \underset{\sim}{=} G_2(\underset{\sim}{A})$$

where "$\underset{\sim}{=}$" means the identity relation between two fuzzy sets.

<u>Remark</u>. It could be shown by simple examples that the identity (8) does not hold in general.

Now, set

(9) $$\vartheta_3(X) = \{G \;;\; G : \mathcal{P}(X) \longrightarrow \mathcal{P}(X),\ \emptyset \neq U \subset X \Longrightarrow G(U) \subset U\} \;,$$

(10) $$\vartheta_4(X) = \{G \;;\; G : \mathcal{P}(X) \longrightarrow \mathcal{P}(X), \emptyset \neq U \subset V \subset X \Longrightarrow U \cap G(V) \subset G(U)\} \;.$$

<u>Assertion 2</u>. Let $G \in \vartheta_3(X)$ and for any $x \in X$ let $G(\{x\}) \neq \emptyset$, $\underset{\sim}{A} \underset{\sim}{\subseteq} X$. Then

(11) $$G_2(\underset{\sim}{A}) \underset{\sim}{\subseteq} G_1(\underset{\sim}{A}) \underset{\sim}{=} \underset{\sim}{A} \;.$$

<u>Assertion 3</u>. Let $G \in \vartheta_3 \cap \vartheta_4$, $\underset{\sim}{A} \subseteq X$, then

(12) $$G_2(\underset{\sim}{A})y = \underset{\sim}{A}y \quad \text{for} \quad y \in \bigcup_{\alpha \in [0,1]} G(\underset{\sim}{A}_\alpha) \;,$$
$$= 0 \quad \text{otherwise} \;.$$

Moreover, the β-level set of the fuzzy set $G_2(\underset{\sim}{A})$ satisfies the following equality:

(13) $$(G_2(\underset{\sim}{A}))_\beta = \bigcup_{\substack{\alpha \in [0,1] \\ \alpha \geq \beta}} G(\underset{\sim}{A}_\alpha) \;.$$

The proofs of Assertions 1 – 3 do not cause serious troubles, they are straightforward.

4. FUZZY OPTIMIZATION PROBLEM

A fuzzy optimization problem is understood as optimization (maximization or minimization) an objective function $g: X \longrightarrow R$ subject to a constraint fuzzy set $\underset{\sim}{U} \lesssim X$. We denote this problem as follows:

(14) $$\max_{\underset{\sim}{U}} g(u) \quad .$$

Immediately arises a question how the optimal solution of (14) should be understood in the above constraint fuzzy set $\underset{\sim}{U}$. To solve this task we define the following set-to-set mapping:

(15) $$G^{o}(U) = \{ x \in X \ ; \ x \in U \ , \ g(x) = \bigvee_{u \in U} g(u) \} \ .$$

We could easily demonstrate that $G^{o} \in \mathcal{V}_3(X) \cap \mathcal{V}_4(X)$, consequently, Assertions 2 and 3 may be applied.

<u>Definition 3 .</u> The optimal solution of the fuzzy optimization problem (14) is a fuzzy set $G_2^{o}(\underset{\sim}{U})$ defined by Definition 2 being applied to the mapping (15) .

<u>Remark .</u> Defining the notion of optimal solution of the fuzzy optimization problem (14) , we use Definition 2 and not Definition 1 . The reason for doing so lies in in Assertion 2 , since $G_1^{o}(\underset{\sim}{U}) \approx \underset{\sim}{U}$. Assertion 3 demonstrates the structure of the optimal solution of (14) .

Usually, the optimal solution of (14) is defined by the other way, see e.g. [2] . Let

(16) $$g(X) = \{\alpha \in R \; ; \; \alpha = g(x) \; , \; x \in X\} \; ,$$

(17) $$\underset{\sim}{U} X = \{\alpha \in R \; ; \; \alpha = \underset{\sim}{U}x \; , \; x \in X\} \; ,$$

and set

$$S = \{(u,v) \in R_2 \; ; \; u \in g(X), \; v \in U X\} \; .$$

The couple $(u_o,v_o) \in S$ is said to be a maximal element of S ,if

$$(u,v) \in S \; , \; u \geqq u_o, v \geqq v_o \quad \text{imply} \quad u = u_o \; , \; v = v_o \; .$$

The set of all maximal elements of S will be denoted by S_{max}.

<u>Definition 4 .</u> By the optimal solution of the fuzzy optimization problem (14) is understood a fuzzy set

(18) $$\int_X \mu(x)/x \quad ,$$

where

(19) $$\mu(x) = \underset{\sim}{U}x \quad \text{for} \quad (g(x),\underset{\sim}{U}x) \in S_{max} \; ,$$
$$= 0 \quad \text{otherwise} \; .$$

The following assertion demonstrates conditions under which the optimal solution of the problem (14) according to Definition 3 coincides with the optimal solution according to Definition 4 .

<u>Assertion 4 .</u> Considering the notation (16)-(19) , it holds

(20) $$\int_X \mu(x)/x \;\underset{\sim}{\subseteq}\; G_2^o(\underset{\sim}{U}) \quad .$$

The opposite inclusion to (20) , i.e.

(21) $$\int_X \mu(x)/x \;\underset{\sim}{\supseteq}\; G_2^o(\underset{\sim}{U})$$

holds if and only if the following condition is valid:

(22) If $x,y \in G^o(\underset{\sim}{U}_\alpha)$ for some $0 < \alpha \leqq 1$, then $\underset{\sim}{U}x = \underset{\sim}{U}y$.

<u>Remark .</u> If for every $0 < \alpha \leqq 1$ g attains its maximum on $\underset{\sim}{U}_\alpha$ in one point at the most, then the statement (22) is true.

<u>Remark .</u> The set-to-set mapping G^{o} defined by (15) is a special element of the system of mappings $\{G^{\varepsilon}\}$, where $\varepsilon \geqq 0$,

$$G^{\varepsilon}(U) = \{ x \in X \; ; \; x \in U \; , \; g(x) \geqq \bigvee_{u \in U} g(u) - \varepsilon \} \; .$$

The extension $G_2^{\varepsilon}(\underset{\sim}{U})$ could be taken as ε -optimal solution of the fuzzy optimization problem (14) .

REFERENCES

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[3] Ramík,J.:"Extension Principle and Fuzzy Mathematical Programming",Kybernetika,19,1983