Linear Optimization with Fuzzy Constraints

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The linear programming problem with constraints with fuzzy coefficients and "ordinary" variables is considered. The inequality relation between two fuzzy numbers is examined. The presented approach is illustrated by a simple numerical example.

1. Introduction and Preliminaries

The paper deals with considering imprecision in optimization problems by means of fuzzy numbers. Approaches usually applied to this situstion may be classified into two categories. The first one consists in weakening the constraints of the classical optimization problem and reformulating it into a new one with the set of feasible solution being fuzzy. The concept of the optimal solution is then based on the intersection of fuzzy subsets. Such an approach is called the "flexible programming" in a survey by Negoita [2].

The second approach enables to model problems whose structures are not known exactly by taking into account the imprecision of parameters already in the phase of construction of the model. The problem obtained in this way has fuzzy coefficients, the variables being non-fuzzy. The presented paper is devoted to some problems arising with this approach /called by Negoita the "robust programming"/. Though applicable to more general situations, the approach is presented for the linear programming problem where the constraints with fuzzy coefficients are considered.

A fuzzy number is a convex normalized fuzzy subset of the real line, i.e. a fuzzy set $\underset{\sim}{a}$ of the real line E_1 such that

$$\forall u,v \in E_1 \; \forall w \in [u\,,\,v] \left[\mu_{\underset{\sim}{a}}(w) \geq \min\{\mu_{\underset{\sim}{a}}(u), \mu_{\underset{\sim}{a}}(v)\} \right] ,$$

$$\exists u \in E_1 \left[\mu_{\underset{\sim}{a}}(u) = 1 \right] .$$

Let $\tilde{a}_1, \ldots, \tilde{a}_n$ be fuzzy numbers, $x \in E_n$. The Extension Principle enables to define the fuzzy number $\tilde{a}_1 x_1 \oplus \ldots \oplus \tilde{a}_n x_n$ /see e.g. [3] /. The problem considered in this paper may be formulated now as follows: Maximize the real function

$$z = c_1 x_1 + \ldots + c_n x_n$$

subject to

$$\tilde{a}_{i1} x_1 \oplus \tilde{a}_{i2} x_2 \oplus \ldots \oplus \tilde{a}_{in} x_n \precsim \tilde{b}_i \quad , \; i = 1,2,\ldots,m$$

$$x_j \geq 0 \quad , \; j = 1,2,\ldots,n,$$

with $c \in E_n$ and fuzzy numbers $\tilde{a}_{ij}$, $\tilde{b}_i$ being given, $x \in E_n$, "$\precsim$" being the sign of inequality between two fuzzy numbers. The crucial problem, of course, is the introduction of a suitable inequality relation on the set of fuzzy numbers.

2. Comparison of fuzzy numbers

Numerous approaches has been developed for ranking fuzzy numbers /see the comprehensive paper [1] /. For the application to optimization problems, the following definition seems to be suitable /see [3]/.

Definition 1. Let $\tilde{a}$, $\tilde{b}$ be fuzzy numbers. Then $\tilde{a} \precsim \tilde{b}$, iff

$$\forall u \in E_1 \, \exists v \in E_1 \left[u \leq v \;\&\; \mu_{\tilde{a}}(u) \leq \mu_{\tilde{b}}(v) \right].$$

To obtain an explicit and useful criterion for the validity of the relation $\tilde{a} \precsim \tilde{b}$, we restrict ourselves to a special class of fuzzy numbers specified in the following definition.

Definition 2. Let $R : [0,+\infty) \to [0,1]$, $R(0) = 1$, be a non-increasing function which is not constant on $[0,+\infty)$. By $\mathfrak{M}_R$ we denote the set of all fuzzy numbers $\tilde{a}$ membership functions of which have the following property:

There are real numbers $m \in E_1$, $\omega \geq 0$ such that

$$\mu_{\tilde{a}}(t) = R\left(\frac{t-m}{\omega}\right) \quad \text{for} \quad t \geq m,$$

$$\mu_{\tilde{a}}(t) = \varphi(t) \quad \text{for} \quad t < m,$$

$\varphi : E_1 \to [0,1]$ being some function non-decreasing on $(-\infty, m)$. A number $\tilde{a} \in \mathfrak{M}_R$ will be called R-fuzzy number.

The following assertion presents a criterion for the inequality relation between two R-fuzzy numbers.

Assertion 1. Let $\underset{\sim}{a}, \underset{\sim}{b} \in \mathfrak{M}_R$, $\underset{\sim}{a} \equiv (\varphi, m, \alpha)_R$, $\underset{\sim}{b} \equiv (\varphi, n, \beta)_R$. Then

$$\underset{\sim}{a} \precsim \underset{\sim}{b}$$

if and only if

$$\varepsilon_R(\alpha - \beta) \leq n - m ,$$

and

$$\delta_R(\alpha - \beta) \leq n - m ,$$

with ε_R, δ_R being defined by the formulas

$$\varepsilon_R = \sup \{ u \, ; \, R(u) = R(0) = 1 \} ,$$
$$\delta_R = \inf \{ u; \, u \geq 0, \, R(u) = \lim_{s \to +\infty} R(s) \} .$$

Remark. As it is usual, we take

$$a \, . \, (\pm \infty) = \pm \infty \quad \text{for} \quad a \geq 0, \quad a \, . \, (\pm \infty) = \mp \infty \quad \text{for } a < 0,$$
$$0 \, . \, (\pm \infty) = 0.$$

To illustrate the application of the Assertion 2 to the linear programming problem formulated in the section 1, a simple numerical example is given in the following section.

3. Numerical example

Set

$$R(u) = \max \{ 0, 1-u \} \quad \text{for} \quad u \geq 0.$$

Then $\varepsilon_R = 0$, $\delta_R = 1$.

Consider the linear programming problem with constraints with fuzzy coefficients belonging to the set $\mathfrak{M}_R$:

Maximize

$$z = 5x_1 + 3x_2$$

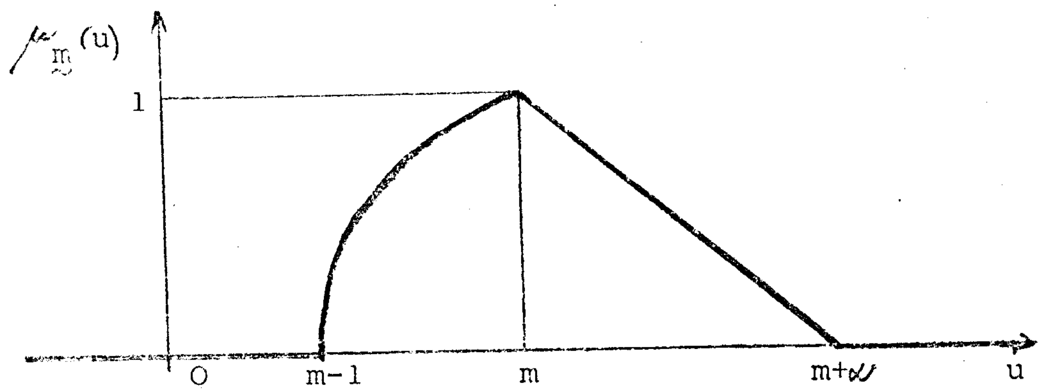
subject to $\qquad\qquad\qquad\qquad$ (P)

$$(\varphi_{(4)}, 4, 1) x_1 \oplus (\varphi_{(5)}, 5, 1) x_2 \precsim (\varphi_{(24)}, 24, 3)$$
$$(\varphi_{(4)}, 4, 2) x_1 \oplus (\varphi_{(1)}, 1, 1) x_2 \precsim (\varphi_{(12)}, 12, 2)$$
$$x_1, x_2 \succsim 0 ,$$

$\varphi_{(m)}$ being for $m \in E_1$ defined as follows:

$$\varphi_{(m)}(u) = \begin{cases} 0 & \text{for } u \in (-\infty, m-1) \\ 1-(u-m)^2 & \text{for } u \in [m-1, m). \end{cases}$$

A fuzzy number $\underset{\sim}{m} = (\varphi_{(m)}, m, \alpha) \in \mathcal{M}_R$ has the shape demonstrated in the following figure:



Examine the set of all feasible solutions of the problem (P). According to the Assertion 2, the system of constraints of the problem (P) is equivalent to the system of inequalities with non-fuzzy coefficients:

$$\begin{aligned} 4x_1 + 5x_2 &\leq 24 \\ 4x_1 + x_2 &\leq 12 \\ 5x_1 + 6x_2 &\leq 27 \\ 6x_1 + 2x_2 &\leq 14 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Note that the latter system does not depend on the functions φ of fuzzy coefficients of the inequalities in the problem (P).

In this way, the linear programming problem with fuzzy constraints (P) was carried over to the classical linear programming problem with double number of constraints. The solution of this problem is

$$x_1 = \frac{15}{13}, \quad x_2 = \frac{46}{13}, \qquad (*)$$

with the corresponding value of the objective function

$$z = \frac{213}{13} .$$

Note that the linear programming problem with non-fuzzy coefficients given as main values of the fuzzy numbers in (P) has the optimal solution

$$x_1 = \frac{9}{4}, \quad x_2 = 3, \qquad (**)$$

with $$z = \frac{81}{4} > \frac{213}{13} .$$

If the problem (P) represents e.g. the problem of finding the optimal production plan, then the fuzzy per unit consumption coefficients may be based on experts estimates or experience. The solution (*) presents then the production plan that is optimal when all circumstances taken into account by making up these coefficients are considered. The solution (**) gives a better profit but is not so "safe" - all the possibilities less favourable than those represented by the "sharp" main values of the coefficients are left out of account.

References

[1] D. Dubois and H. Prade, Ranking fuzzy numbers in the setting of possibility theory, Information Sciences, 30 /1983/

[2] C. V. Negoita, The current interest in fuzzy optimization, Fuzzy Sets and Systems 6 /1981/

[3] J. Ramík, J. Římánek, Comparison of fuzzy numbers and fuzzy optimization, BUSEFAL, No. 17 /1984/

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