ON SOME BASIC PROBLEMS OF PROPOSITIONAL FUZZY LOGIC

Jozef Šajda, Institute of Technical Cybernetics, Slovak Academy of Sciences, Bratislava

1. Fuzzy logic as a formal system

In this paper, fuzzy logic will be understood as a formal system with a given set of axioms and a set of inference rules. The axioms are fuzzy formulas to be selected in advance, to which one assigns a certain degree of truth. The validity of other formulas will be deduced from the axioms whereby from the truth degree of fuzzy axioms the truth degree of proper fuzzy deductions should be derived. The axiomatical building of fuzzy logic brings many advantages, mainly an automatic mechanism for confirming the validity of fuzzy formulas and determining their truth values derived from the truth values of their premises. Defining fuzzy logic as a formal system means to form a syntactical system $\langle F,X,R \rangle$ in the set F of all fuzzy formulas as a fuzzy theory in such a way, that one proceeds from a chosen fuzzy set $X \subsetneq F$ of logical plus extralogical axioms, from which, by means of a chosen set R of many-valued inference rules, all possible fuzzy consequences of the set X are proved syntactically. Such formal procedure of proving formulas from premises has a mechanical character, so that its performance may be entrusted to the computer.

On the other hand, it is possible to conceive fuzzy logic as a semantical system $\langle F,S \rangle$ in such a way, that in the set F one defines a semantic S as a certain set of truth functions $T: F \to W$ - where W denotes a set of truth values - together with an operation of semantical following.

Such a conception of fuzzy logic implies the cardinal problem of completeness: Is it possible to choose a fuzzy set A of axioms and a set R of inference rules in such a way, that the degree of provability of a given formula in the syntactical system $\langle F,A,R \rangle$ will be identical to its following - degree in the semantical system $\langle F,S \rangle$?

In cases when the answer to so formulated question is affirmative, we say that such a fuzzy logic is complete. The solving of the completeness problem is of a great both theoretical and practical importance, because syntactical inference algorithms are then strong enough to deduce semantically correct propositions.

2. Algebraic base of fuzzy logic

Some time ago, Goguen [2] proposed to build up Zadeh's theory of fuzzy sets [1] on the basis of lattice structures. It appeared, however, to be necessary to spread these structures with semigroup operations $\otimes$ of multiplication, and $\rightarrow$ of residuation. These operations appeared to be adequate tools to logical operations of context and implication. The algebraic structure completed in such a way was called a /complete/ residuated lattice, $\mathbb{L} = \langle L, \otimes, \rightarrow \rangle$ where the carrier L is the original /complete/ lattice of truth values.

The operations $\otimes$ and $\rightarrow$ are in the lattice $\mathbb{L}$ mutually connected by the relation

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c$$

where a,b,c are arbitrary elements of the lattice L, and $\leq$ is a partially-ordering relation in L.

For purposes of fuzzy logic it appears to be useful to introduce into residuated lattices other operations appropriate for axiomatizing the fuzzy logic. Thus the so called fitting operations G_d are defined together with the function Ar(d) of arity of the G_d, and the function Ex(d) of exponents of G_d expressed by $\langle k_1, k_2, \dots, k_{Ar(d)} \rangle$ of natural numbers, where $d \in \Delta$, and Δ is an arbitrary set.

Let $\mathbb{L} = \langle L, \otimes, \rightarrow \rangle$ be residuated lattice and $\mathcal{G} = \{G_d, d \in \Delta\}$ be a set of operations on L such that every operation G_d is Ar(d)- ary and fits the lattice $\mathbb{L}$ by the exponent chain Ex(d). Then $\mathbb{E} = \langle \mathbb{L}, \mathcal{G} \rangle$ is said to be an enriched residuated lattice of the type $\langle Ar, Ex \rangle$.

As the most important lattices of this kind are to be regarded

1. the (m+1)- element chain $\mathbb{L} = \langle C_{m+1}, \otimes, \longrightarrow \rangle$, where $C_{m+1} = \{0 = a_0 < a_1 < \cdots < a_m = 1\}$, $m \geq 1$
2. the Łukasiewicz interval $\mathbb{Ł} = \langle I, \oplus, \rightarrow \rangle$, where $I = \langle 0,1 \rangle$, the multiplication $\oplus$ is defined by the relation
   $a \oplus b = \max(0, a+b-1) = 0 \vee (a+b-1)$
   and residuation $\rightarrow$ in turn, by
   $a \rightarrow b = \min(1, 1-a+b) = 1 \wedge (1-a+b)$.

In his papers [3], [4], [5] Pavelka has proved that only in these two cases is it possible to axiomatize the fuzzy logic.

3. Syntax and semantics of propositional fuzzy logic

It seems to be quite natural to choose as the truth value of a given formula $\varphi \in F$ just the degree of its membership $f\varphi$ to the fuzzy subset $f: F \longrightarrow \langle 0,1 \rangle$. In such a conception one uses in fuzzy logic the following four elementary operations:

- negation $\neg$, for which $T(\neg\varphi) = 1 - T(\varphi)$
- conjunction $\wedge$, for which $T(\varphi \wedge \psi) = \min\{T(\varphi), T(\psi)\}$
- disjunction $\vee$, for which $T(\varphi \vee \psi) = \max\{T(\varphi), T(\psi)\}$
- implication $\Rightarrow$, which can be regarded as derivable from preceding operations. Of course, this implies also derivability of the truth value $T(\varphi \Rightarrow \psi)$.

The truth value $T(\varphi \Rightarrow \psi)$ can not be chosen optionally, but only in connection with another operations of fuzzy logic. The solution of problems like this appeared to require the introduction into the set W of truth values, just the structure of the enriched residuated lattice $\mathbb{E} = \langle\langle L, \otimes, \rightarrow\rangle, \mathcal{G}\rangle$ mentioned above.

In such structure of propositional fuzzy logic we have to define the following four binary connectives:

- conjunction $\wedge$, interpreted in the residuated lattice by the operation $\wedge$ of meet,
- disjunction $\vee$, interpreted in the residuated lattice by the operation $\vee$ of join,
- implication $\Rightarrow$, interpreted in the residuated lattice by the operation $\rightarrow$ of residuation,
- context $\&$, interpreted in the residuated lattice by the operation $\otimes$ of multiplication.

In this conception, the truth values $a \in L$ are presented as a nullary connective $\underline{a}$, implying that the unary connective $\neg$ of negation is derivable from the connectives $\underline{a}$ and $\Rightarrow$, by the equation $\neg\varphi = \varphi \Rightarrow \underline{0}$. The connective $\&$ of the context enables to define context powers of fuzzy formulas

$$\varphi^n = \underbrace{\varphi \,\&\, \varphi \,\&\, \ldots \,\&\, \varphi}_{\text{n-times}}$$

with a similar meaning as the Zadeh's operators of the type "very".

The meaning of the connective $\&$ of the context may be illustrated by the tautology

$$((\varphi \,\&\, \psi) \Rightarrow \chi) \Leftrightarrow (\varphi \Rightarrow (\psi \Rightarrow \chi))$$

where the biimplication $\Leftrightarrow$ is a derived logical connective defined by the formula

$$\varphi \Leftrightarrow \psi = (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi).$$

As the inference rules of the type $r = \langle r', r'' \rangle$ we choose the following rules:

R1. the modus ponens rule:

$$r_1': \frac{\varphi,\ \varphi \Rightarrow \psi}{\psi}, \qquad r_1'': \frac{a, b}{a \otimes b}$$

R2. the lifting rule:

$$r_2'a: \frac{\varphi}{\underline{a} \Rightarrow \varphi}, \qquad r_2''a: \frac{b}{a \rightarrow b}$$

R3. the elimination rule

$$r_3'a: \frac{\varphi \vee \underline{a}}{\varphi}, \qquad r_3''a: \frac{b}{a \leftarrow b}$$

R4. the consistency-testing rule:

$$r_4'a(\underline{a}) = 0, \qquad r_4''a(b) = \begin{cases} 0 & \text{if } b \not\leq a \\ 1 & \text{otherwise} \end{cases}$$

Pavelka has proved [5] that all these inference rules on $F(P,L,\Delta)$ are sound with respect to the semantics $S(P,\mathbb{E})$ defined for the enriched residuated lattice $\mathbb{E}$ and the set P of propositional variables.

As a set A of axioms we choose a set of 25 fuzzy formulas, most of them are known to be axioms of non-fuzzy logic. These form tautologies of propositional fuzzy logic, i.e. formulas with the tautological degree $(C_{S(P,\mathbb{E})}\ \underline{0})\varphi = 1$. The tautological degree of a formula φ presents a measure of simantical following of a fuzzy consequent φ from an empty set of extralogical axioms. The symbol $C_{S(P,\mathbb{E})}$ represents a consequence operation in the semantics $S(P,\mathbb{E})$, which assigns to any fuzzy set $X: F \rightarrow L$ of premises a fuzzy set $C_{S(P,\mathbb{E})}X$ of consequences. For a given formula $\varphi \in F$ the degree of following is represented by the expression

$(C_{S(P,\mathbb{E})}X)\varphi = \bigwedge \{ T\varphi \mid T \in S(P,\mathbb{E}),\ T\psi \geq X\varphi$ for all $\psi \in F(P,L,\Delta) \}$,

where T denotes a truth function.

Pavelka has proved [5] that syntax $\langle A_1, R_1 \rangle$ - where A_1 is a certain subset of 23 axioms of the set A, and $R_1 = \{R2, R3, R4\}$ - is semantically complete for any (m+1) - element chain C_{m+1}. Analogically, the syntax $\langle A_2, R_2 \rangle$ - where A_2 is a certain subset of 24 axioms of the set A and $R_2 = \{R1, R2\}$ -is semantically complete for the Łukasiewicz interval $Ł = \langle \langle 0,1 \rangle , \oplus , \longrightarrow \rangle$.

<u>References:</u>

[1] Zadeh,L.A.: Fuzzy Sets. Information and Control, 8 /1965/ 348-353.

[2] Goguen,J.A.: L-fuzzy sets. Journal of Math. Analysis and its Applications, 18 /1967/, 145-174.

[3] Pavelka,J.: On fuzzy logic I: Many-valued rules of inference. Zeitschrift für math. Logik und Grundlagen der Mathematik, 25, /1979/, 45-52.

[4] Pavelka,J.: On fuzzy logic II: Enriched residuated lattices and semantics of propositional calculi. Zeitschrift für math. Logik und Grundlagen der Mathematik, 25, /1979/, 119-134.

[5] Pavelka,J.: On fuzzy logic III: Semantical completeness of some many-valued propositional calculi. Zeitschrift für math. Logik und Grundlagen der Mathematik, 25 /1979/, 447-464.