Lower Solutions of Systems of Fuzzy Equations

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Abstract: Lower Solutions of systems of fuzzy equations are determined by lower solutions of component equations.

1. Introduction

Fuzzy equations with extended operations have been defined and studied by Sanchez[2]. Gottwald[1] has shown the sufficient and necessary condition for the existence of solutions of systems of fuzzy equations with a binary operation defined on a universe of discourse.

Here we present a method to determine lower solutions of systems of fuzzy equations by lower ones of component equations satisfying certain conditions.

2. Preliminaries

We recapitulate some underlying definitions and results by Sanchez[2] and Gottwald[1].

Let $\mathbb{X} = \{x\}$ be a universe of discourse with a binary operation $*$, and $I(\mathbb{X})$ be the set of all fuzzy sets on $\mathbb{X}$.

Then we have two binary operations $*$ and $\tilde{*}$ on $I(\mathbb{X})$, which are defined as follows:

$$A*B(z) \triangleq \sup_{x*y=z} A(x) \wedge B(y) \quad ,$$

$$C\tilde{*}D(y) \triangleq \inf_{x*y=z} D(x) \,\alpha\, C(z) \quad ,$$

where $A,B,C,D \in I(\mathbb{X})$, and $y,z \in \mathbb{X}$, and $\wedge$ and α denote the min and an operator defined by Sanchez[3], respectively.

Sanchez also proved the distributivity of $*$ over the union of fuzzy sets, i.e.,

(1) $A * (B \cup C) = (A * B) \cup (A * C)$.

Now let us consider a system of fuzzy equations

(2) $A_i * X = C_i$, $i=1,\ldots,h$.

Defining

$\mathcal{X} \triangleq \{X \mid A_i * X = C_i , i=1,\ldots,h \}$, and

$\mathcal{X}_i \triangleq \{X \mid A_i * X = C_i \}$ for each i ,

we have

(3) $\mathcal{X} = \bigcap_{i=1,h} \mathcal{X}_i$

and the following two theorems.

Theorem 1[2].

$\mathcal{X}_i \neq \phi \leftrightarrow C_i \tilde{*} A_i \in \mathcal{X}_i$, and then $C_i \tilde{*} A_i$ is the greatest element in $\mathcal{X}_i$.

Theorem 2[1].

$\mathcal{X} \neq \phi \leftrightarrow \bigcap_{i=1,h} (C_i \tilde{*} A_i) \in \mathcal{X}$ and then $\bigcap_{i=1,h} (C_i \tilde{*} A_i)$ is the greatest element in $\mathcal{X}$.

Now we conclude this preliminary section with a definition.

Definition.

Let $\mathcal{A} \subset I(\mathbb{X})$. $P \in \mathcal{A}$ is called a lower element in $\mathcal{A}$ if $P' \leq P$ for $P' \in \mathcal{A}$ implies $P' = P$,

where $\leq$ denotes inclusion in the sense of fuzzy sets theory.

3. Lower solutions of systems of fuzzy equations

Here we determine lower solutions, which mean lower elements in X, of the system (2) by those of each component equations under certain conditions.

Defining

$X^0 \triangleq$ the set of all lower elements in X, and

$X_i^0 \triangleq$ the set of all lower elements in X_i ,

we assume the following two conditions throughout this section:

(*-1) each X_i has lower elements in X_i , and

(*-2) $x \in X_i \leftrightarrow \exists x_0 \in X_i^0$ such that $x_0 \le x \le C_i \tilde{*} A_i$.

In order to avoid trivial cases, we assume also

$X_i \ne \phi$ for each i .

Now we get the following two lemmas.

Lemma 1.

Let $P_i \in X_i$ for each i.

Then $\bigcup_{i=1,h} P_i \in X \leftrightarrow A_i * P_j \le C_i$ for each i and $j \ne i$.

Proof. $\bigcup_{t=1,h} P_t \in X \leftrightarrow A_i * (\bigcup_{t=1,h} P_t) = \bigcup_{t=1,h} A_i * P_t = C_i$ (from (1))

$\leftrightarrow C_i \cup (\bigcup_{t=1,h,\ t\ne i} A_i * P_t) = C_i$ (from $P_i \in X_i$) $\leftrightarrow \bigcup_{t=1,h,\ t\ne i} A_i * P_t \le C_i \leftrightarrow$

$A_i * P_t \le C_i$ for each i and $t \ne i$. Q.E.D.

Lemma 2.

Let $P_i \in X_i$ for each i.

Then, $\bigcup_{i=1,h} P_i \in X \leftrightarrow \bigcap_{i=1,h} (C_i \tilde{*} A_i) \in X$ and $\bigcup_{i=1,h} P_i \le \bigcap_{i=1,h} (C_i \tilde{*} A_i)$.

Proof. The if part is a direct consequence of Theorem 2.

The only if part: From the assumption, we have

$$P_t \le C_i * A_i \quad \text{for} \quad 1 \le t, i \le h.$$

Hence $A_i * P_t \le A_i * (C_i \tilde{*} A_i) = C_i$ ($t \ne i$), because $P_t \in X_t$, i.e.,

$C_t \tilde{*} A_t \in X_t$ and Eq.(1) means that $A * B \le A * C$ for $B \le C$.

Consequently, from Lemma 1, we have $\bigcup_{i=1,h} P_i \in X$. Q.E.D.

Defining $\tilde{P}_i \triangleq C_i \tilde{*} A_i$ and $\tilde{P} \triangleq \bigcap_{i=1,h} \tilde{P}_i$ ($= \bigcap_{i=1,h} (C_i \tilde{*} A_i)$) for brevity,

we have

$X \ne \phi \leftrightarrow \tilde{P} \in X \leftrightarrow \tilde{P} \in X_i$ for each i (from (3) and Theorem 2), and

${}^{\exists}P_i^0 \in X_i^0$ for each i such that $P_i^0 \le \tilde{P} \le \tilde{P}_i$ (from (*-2)).

Therefore, defining

$$X_i^0(\tilde{P}) \triangleq \{ P_i^0 \in X_i^0 \mid P_i^0 \le \tilde{P} \} ,$$

$$\underline{X}^0(\tilde{P}) \triangleq \{ \bigcup_{i=1,h} P_i^0 \mid P_i^0 \in X_i^0(\tilde{P}) \} , \text{ and}$$

$X^0(\tilde{P}) \triangleq$ the set of all lower elements in $\underline{X}^0(\tilde{P})$,

and assuming that $X \ne \phi$, we have

$\underline{X}^0(\tilde{P}) \ne \phi$, $X^0(\tilde{P}) \ne \phi$, and $X^0(\tilde{P}) \subseteqq \underline{X}^0(\tilde{P}) \subseteqq X$ (from the definitions of $X^0(\tilde{P})$ and $\underline{X}^0(\tilde{P})$, and Lemma 2).

Then we obtain the following fundamental

Lemma 3.

If $X \ne \phi$, then $X^0 \supseteqq X^0(\tilde{P})$.

Proof. Since $X^0(\tilde{P}) \ne \phi$ from the assumption, there exists a $P^0 \in X^0(\tilde{P})$, and so $\tilde{P} \ge P^0$. Let us assume that ${}^{\exists}P \in X$ such that $P^0 \ge P$.

From (*-2), ${}^{\exists}P_i^0 \in X_i^0$ for each i such that $\tilde{P} \ge P \ge P_i^0$.

Hence $\tilde{P} \ge P \ge \bigcup_{i=1,h} P_i^0$ and $\bigcup_{i=1,h} P_i^0 \in \underline{X}^0(\tilde{P})$ ($\subset X$).

Setting $P^{0'} \triangleq \underset{i=1,h}{\cup} P_i^0$, from the definition of $X^0(\tilde{P})$, only the following two cases are possible:

(a) $P^{0'} \varepsilon X^0(\tilde{P})$,

or (b) $P^{0'} \not\varepsilon X^0(\tilde{P})$,i.e., $^{\exists}P^{0''} \varepsilon X^0(\tilde{P})$ such that $P^{0''} < P^{0'}$.

Since $P^0 \geq P \geq P^{0'} > P^{0''}$ and $P^{0''}, P^0 \varepsilon X^0(\tilde{P})$, the case (b) leads us to a contradiction. For the case (a), it is impossible to be $P^0 > P^{0'}$ by the same reason as the case (b).

Whence, we have $P^0 = P^{0'}$ and so $P^0 = P$.

This means that P^0 is a lower element in X ,i.e., $P^0 \varepsilon X^0$. Q.E.D.

This lemma shows the existence of lower solutions of the system (2) under the assumtions (*-1), (*-2) and $X \neq \phi$, and this lemma also leads us to the following

Theorem 3.

If a system (2) of fuzzy equations satisfies (*-1) and (*-2), and if it has a solution, then $X^0 = X^0(\tilde{P})$.

Proof. By Lemma 3, it suffices to show $X^0 \subseteqq X^0(\tilde{P})$. Since $X^0 \neq \phi$ from the assumptions and Lemma 3, there exists a lower element in X , i.e., $^{\exists}P^0 \varepsilon X^0$.

Hence, $P^0 \varepsilon X_i^0$ for each i (from (3)) $\rightarrow$ For each i, $^{\exists}P_i^0 \varepsilon X_i^0$ such that $\tilde{P}_i \geq P^0 \geq P_i^0 \rightarrow \tilde{P} \geq P^0 \geq \underset{i=1,h}{\cup} P_i^0 \rightarrow \underset{i=1,h}{\cup} P_i^0 \varepsilon X$(from Lemma 2) and $P^0 = \underset{i=1,h}{\cup} P_i^0$ (from $P^0 \varepsilon X^0$). Therefore, since $\underset{i=1,h}{\cup} P_i^0 \varepsilon \underline{X}^0(\tilde{P})$, $P^0 \varepsilon X^0$, and $X^0(\tilde{P}) \subseteqq X$, we have $P^0 = \underset{i=1,h}{\cup} P_i^0 \varepsilon X^0(\tilde{P})$.

Consequently, $X^0 \subseteqq X^0(\tilde{P})$, as was to be proved. Q.E.D.

This theorem gives us a method to determine lower solutions in X by those in each X_i.

Finally we note that all these proofs of lemmas and theorems in this paper are available without substantial changes for systems of fuzzy relational equations with triangular norms and the same result as Theorem 3 will be obtained immediately.

References

[1] Gottwald,S., On the existence of solutions of systems of fuzzy equations, Fuzzy Sets and Systems, 12(1984), 301-302.

[2] Sanchez,E., Solution of fuzzy equations with extended operations, ibid., 12(1984), 237-248.

[3] Sanchez,E., Resolution of composite fuzzy relation equations, Information and Control, 30(1976), 38-48.