THE STABILITY OF SOLUTIONS OF FUZZY RELATION EQUATIONS
Part 1 : Directional Perturbation

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In this paper, we advance a new idea fuzzy perturbation, and define perturbation matrix. The stability of the solutions of fuzzy relation equations and the generalized solution of the unsolvable equation are defined by means of the matrix. By use of the concepts we advance the programs of solving the two open problems in the inverse problem of fuzzy multifactorial evaluation.

Keywords : Fuzzy perturbation matrix, Stability of the solutions, Generalized solution, Fuzzy relation equation, Fuzzy mutifactorial evaluation and its inverse problem.

1. Introduction

Let $U=\{u_1,\ldots,u_n\}$ be a set of factors and $V=\{v_1,\ldots,v_m\}$ be a set of evaluations, and $F(U)$, $F(V)$ and $F(U\times V)$ is the family of all fuzzy sets on U, V, and $U\times V$, respectively. If $B\in F(V)$ and $R\in F(U\times V)$ are given, the inverse problem of this fuzzy multifactorial evaluation is expressed as a fuzzy relation equation which regrads $X\in F(U)$ as an unknown element:

$$X\circ R=B \tag{1.1}$$

i.e. $\bigvee_{1\leqslant i\leqslant n}(x_i\wedge r_{ij})=b_j \qquad j=1,\ldots,m$

where $X=(x_1,\ldots,x_n)$, $R=(r_{ij})_{n\times m}$ and $B=(b_1,\ldots,b_m)$.

Let $F(.)$ be the family of all fuzzy sets on any set. We define a partially ordering "$\leqslant$" in $F(.)$: $A_1\leqslant A_2$ iff $A_1\subset A_2$ for any A_1, $A_2\in F(.)$. In addition, it is denoted $A_1<A_2$ that $A_1\leqslant A_2$ and $A_1\neq A_2$.

Put $\mathfrak{X}=\{X\in F(U)\mid X\circ R=B\}$, it is common knowledge that the partially ordered set $(\mathfrak{X},\leqslant)$ is an infinite semilattice with the greatest element. In the paper, the greatest element of $\mathfrak{X}$ is always denote $G\triangleq(g_1,\ldots,g_n)$.

We advance the two open problems in the inverse problem of fuzzy multifactorial evalution:

(1) How to choose an element X_o from $\mathfrak{X}$ as the distribution of the weight number of the evalution process when $|\mathfrak{X}|>1$?

(2) How to choose $A_0 \in F(U)$ as the distribution of the weight number when $\mathfrak{X}=\emptyset$?

As everyone knows, the membership functions shown fuzzy sets are of elasticity. If we make R a small change, denoted R^{ε}, then R and R^{ε} may not be distinguished in the application. But equation XoR=B and $XoR^{\varepsilon}=B$ may have much distinction. By use of the distinction, we advance an idea,"fuzzy perturbation", so that the two open problems may be solved by the idea.

2. Stability of the solutions

Definition 2.1 Let

$$\lambda_{ij}=\begin{cases}1, & r_{ij}\geq b_j \\ 0, & r_{ij}<b_j\end{cases} \qquad i=1,\ldots,n,\ j=1,\ldots,m \qquad (2.1)$$

Boolean matrix $\lambda=(\lambda_{ij})$ is called the characteristic matrix of the equation (1.1) and $\mu_j=\sum_{i=1}^{n}\lambda_{ij}$ is called a characteristic number.

Proposition 2.1 $\mathfrak{X}\neq\emptyset \Longrightarrow \mu_j\geq 1$, $j=1,\ldots,m$. ▮

Definition 2.2 Let $\varepsilon\in(0,1]$, $R^{\varepsilon}=(r_{ij}-a_{ij}\varepsilon)_{n\times m}$ is called a ε-perturbation matrix of R. Where

$$a_{ij}=\begin{cases}0, & \lambda_{ij}=0 \\ \varphi(r_{ij}), & \lambda_{ij}=1 \text{ and } r_{ij}\neq 0 \\ |o(\varepsilon)|, & \lambda_{ij}=1 \text{ and } r_{ij}=0\end{cases} \qquad (2.2)$$

$o(\varepsilon)$ is a higher order infinitesimal of ε ; $\varphi=\eta\varphi_1+(1-\eta)\varphi_2$, $\eta\in[0,1]$; φ_1 and $\varphi_2\in(0,1]^{(0,1]}$ and satisfy conditions: φ_1 is strictly monotone increasing on $(o,1]$, φ_2 is strictly monotone increasing on $(0,1/2]$ and strictly monotone decreasing on $(1/2,1]$.

Remark When $\lambda_{ij}=1$ and $r_{ij}=0$, $r_{ij}-a_{ij}\varepsilon<0$, i.e. $R^{\varepsilon}\notin F(U)$. But this is of no importance.

Definition 2.3 Let $X\in\mathfrak{X}$, X is called ε-stable if the following equation is fulfilled:

$$XoR^{\varepsilon}=B \qquad (2.3)$$

else, ε-unstable. The set of all ε-stable solutions is denoted $\mathfrak{X}^{\varepsilon}$.

Definition 2.4 For any $X\in\mathfrak{X}$, put $W(X)=\{\varepsilon\in(0,1] \mid XoR^{\varepsilon}=B\}$. Take the mapping $S:\mathfrak{X}\longrightarrow[0,1]$ such that

$$S(X)=\begin{cases}\sup W(X), & W(X)\neq\emptyset \\ 0, & W(X)=\emptyset\end{cases} \qquad (2.4)$$

$S(X)$ is called degree of stability of X, about the equation (1.1)

Proposition 2.2 S is a order-preserving mapping. ▮

Corollary The greatest element of $\mathfrak{X}$ is of the greatest degree of stability, and a certain minimal element of $\mathfrak{X}$, of the least.

Definition 2.5 For any $X\in\mathfrak{X}$ and $j\in\{1,\ldots,m\}$, put $T_j=\{t \mid x_t\wedge r_{tj}=b_j\}$. For any $t\in T_j$, put $W_{tj}(X)=\{\varepsilon\in(0,1] \mid x_t\wedge(r_{tj}-a_{tj}\varepsilon)=b_j\}$. Take mapping $S_{tj}:\mathfrak{X}\longrightarrow[0,1]$ such that

$$S_{tj}(X)=\begin{cases}\sup W_{tj}(X), & W_{tj}(X)\neq\emptyset\\ 0, & W_{tj}(X)=\emptyset\end{cases}$$

$S_{tj}(X)$ is called degree of subpart stability of X, about the equation (1.1). In addition, we take mapping $S_j:\mathfrak{X}\longrightarrow[0,1]$ such that

$$S_j(X)=\bigvee_{t\in T_j} S_{tj}(X)$$

$S_j(X)$ is called degree of part stability of X, about equation (1.1).

Proposition 2.3 For any $X\in\mathfrak{X}$, $S(X)=\bigwedge_{1\leq j\leq m} S_j(X)$, consequently, $S(X)=\bigwedge_{1\leq j\leq m}\left(\bigvee_{t\in T_j} S_{tj}(X)\right)$. ▮

3. Ordered quotient set $(\mathfrak{X}/S,\dashv)$ and its meaning

An equivalence "$\sim$" on $\mathfrak{X}$ may be determined by the mapping S: $X_1\sim X_2$ iff $S(X_1)=S(X_2)$ for any $X_1,X_2\in\mathfrak{X}$. Hence we obtain a quotient set of $\mathfrak{X}$: $\mathfrak{X}/S=\{\overline{X} \mid X\in\mathfrak{X}\}$, where $\overline{X}$ is an equivalence class for X. Besides, $\mathfrak{X}/S$ is ordered by "$\dashv$": $\overline{X}_1\dashv\overline{X}_2$ iff $S(X_1)\leq S(X_2)$ for any $\overline{X}_1$, $\overline{X}_2\in\mathfrak{X}/S$.

Theorem 3.1 $(\mathfrak{X}/S,\dashv)$ is a finite chain. ▮

According to theorem 3.1, we can suppose $\mathfrak{X}/S=\{\overline{X}_1,\overline{X}_2,\ldots,\overline{X}_q\}$, where $\overline{X}_q\dashv\overline{X}_{q-1}\dashv\cdots\cdots\dashv\overline{X}_1$.

Let L_i be the set of all minimal element of $\overline{X}_i$, $Q_i=\bigvee_{X\in L_i} X$ and $\mathfrak{X}_i=\{X\in F(U) \mid Q_i\leq X\leq G\}$, it is easy to obtain the following

Proposition 3.1 $\mathfrak{X}_i\subset\bigcup_{j=1}^{i}\overline{X}_j$, for $i=1,\ldots,q$. ▮

Definition 3.1 Let $(P,\leq)$ be a nonempty ordered set and P_* be the set of all minimal element of P. P is of minimal character if $\forall a\in P$ $\exists b\in P_*$, such that $b\leq a$.

ALGORITHM OF CHOOSING THE DISTRIBUTION OF THE WEIGHT NUMBERS X_o

Let $\overline{X}_i$ be of minimal character for $i=1,\dots,q$.

First: Form $\mathcal{X}_0$ which is the range of choosing X_o.

(1) If $Q_1 < G$, take $\mathcal{X}_0 = \mathcal{X}_1$. Else

(2) If $Q_2 < G$, take $\mathcal{X}_0 = \mathcal{X}_2$. Else

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(q-1) If $Q_{q-1} < G$, take $\mathcal{X}_0 = \mathcal{X}_{q-1}$. Else

(q) Take $\mathcal{X}_0 = \mathcal{X}_q$.

Second: Determine X_o.

For any $A, D \in F(U)$ and any $a \in [0,1]$, stipulate

$$A+D=(a_1+d_1,\dots,a_n+d_n)$$
$$aA=(aa_1,\dots,aa_n)$$

Since the degree of stability of solutions shows the extent that the solutions depend on the equation, in orther words, it shows the reliable information of the evalution process, hence, X_o should be determined as follows:

$$X_o = \begin{cases} \dfrac{S(G)}{S(G)+S(X_*)}G + \dfrac{S(X_*)}{S(G)+S(X_*)}X_*, & S(G)+S(X_*)>0 \\ G, & \text{else} \end{cases}$$

where X_* is the least element of $\mathcal{X}_0$.

4. Partially ordered structure of $\overline{X}_i$

We will prove that $\overline{X}_i$ is of minimal character without any condition.

A nonempty partially ordered P is called lower inductiue if every chain of P has a lower bound.

Proposition 4.1 That P is lower inductiue imply that P is of minimal character. ∎

Proposition 4.2 $\mathcal{X}$ is lower inductiue, hence $\mathcal{X}$ is of minimal character. ∎

Theorem 4.1 $\overline{X}_i$ is lower inductiue, hence $\overline{X}_i$ is of minimal character, for $i=1,\dots,q$. ∎

5. Generalized solutions of the unsolvable equation

Definition 5.1 Let $\delta \in (0,1]$, $R^{\delta}=(r_{ij}+b_{ij}\delta)_{n\times m}$ is called a δ-perturbation matrix of R, where

$$b_{ij}=\begin{cases}\varphi(r_{ij}) & , \lambda_{ij}=0\\ 0 & , \lambda_{ij}=1\end{cases}$$

and the φ is the same as the φ of definition2.3.

Definition 5.2 Suppose $\mathfrak{X}=\emptyset$ and $X\in F(U)$. X is called a δ-generalized solution of the equation (1.1) if the following equation is fulfilled:

$$X\circ R^{\delta}=B \tag{5.1}$$

The set of all the generalized solutions is denoted $\mathfrak{X}^{\delta}$.

Definition 5.3 δ^{o} is called the deviation of the generalized solutions, about the equation (1.1), if

$$\delta^{o}=\begin{cases}\inf W & , W\neq\emptyset\\ 1 & , W\neq\emptyset\end{cases}$$

where $W=\{\delta\in(0,1]\mid \mathfrak{X}^{\delta}\neq\emptyset\}$.

Definition 5.4 Let $\mathfrak{X}^{\delta}_{ij}$ be the set of all the solutions of the following equation

$$x_i\wedge(r_{ij}+b_{ij}\delta)=b_j$$

and $W_{ij}=\{\delta\in(0,1]\mid \mathfrak{X}^{\delta}_{ij}\neq\emptyset\}$. δ^{o}_{ij} is called the subpart deviation of the generalized solutions, about the equation (1.1), if

$$\delta^{o}_{ij}=\begin{cases}\inf W_{ij} & , W_{ij}\neq\emptyset\\ 1 & , W_{ij}=\emptyset\end{cases}$$

Besides $\delta^{o}_{j}=\bigwedge_{1\leq i\leq n}\delta^{o}_{ij}$ is called the partial deviation of the generalized solutions, about the equation (1.1).

Proposition 5.1 $\delta^{o}=\bigvee_{1\leq j\leq m}\delta^{o}_{j}$, hence $\delta^{o}=\bigvee_{1\leq j\leq m}(\bigwedge_{1\leq i\leq n}\delta^{o}_{ij})$. ∎

Definition 5.5 Suppose $\mathfrak{X}\neq\emptyset$. The equation (1.1) is called pesudo-unsolvable if $\delta^{o}=0$.

Definition 5.6 We stipulate the operation of the Boolean matrixs $A=(a_{ij})$ and $D=(d_{ij})$ "+": $A+D=(a_{ij}+d_{ij})$. When $\mathfrak{X}\neq\emptyset$ the characteristic matrix λ has the resolution as follows:

$$\lambda=\lambda_1+\lambda_2+\ldots\ldots+\lambda_r \tag{5.2}$$

where $\lambda_k\triangleq(\lambda^{k}_{ij})$ satisfies the condition $\mu^{k}_{j}=1$, and μ^{k}_{j} is a characteristic number of λ_k. The formula (5.2) is called the perfect resolution of λ. Besides, the index set $I_k\triangleq\{i_s\mid \lambda^{k}_{i_s,s}=1,\ s\in\{1,\ldots,m\}\}$ is called a place of λ_k, and $I\triangleq\{I_k\}$ is called the place family of λ .

Remark. The symbols of the above concepts which are determined by the characteristic matrix λ^{δ} of R^{δ} should be load with δ , for example, I_k should be denoted I^{δ}_{k}.

Priposition 5.2 (1) $\forall\ X\in\mathcal{X}\ \exists I_k=\{i_1,\ldots,i_m\}\in I$ such that

$$x_{i_s}\wedge r_{i_s,s}=b_s \qquad s=1,\ldots,m \tag{5.3}$$

The formula (5.3) will be denoted $I_k=I_k(X)$.

(2) For any $X,Y\in\mathcal{X}$ and $X\leqslant Y$, if $\exists I_k\in I$ such that $I_k=I_k(X)$, then $I_k=I_k(Y)$. ∎

6. Estimation of degree of M-approximation

Let $A=(a_1,\ldots,a_m),D=(d_1,\ldots,d_m)\in F(V)$, we call

$$d_p(A,D)\triangleq\Big(\sum_{j=1}^{m}|a_j-d_j|^p\Big)^{1/p}, \qquad p\geqslant 1$$

the distance between A and D, and call $\|A\|_p=d_p(A,\emptyset)$ the norm of A.

Definition 6.1 Suppose $\mathcal{X}^\delta\neq\emptyset$. For any $X\in\mathcal{X}^\delta$ we call

$$\|XoR-B\|_p=d_p(XoR,B)$$

the degree of M-approximation of X, which is with respect to the equation (1.1).

Theorem 6.1 Suppose $\mathcal{X}^\delta\neq\emptyset$. For any $X\in\mathcal{X}^\delta$ we have

$$\|XoR-B\|_p\leqslant c\delta$$

where c is a constant. ∎

Corollary 1 $c\leqslant\min_{I_k^\delta(X)\in I^\delta}\Big(\big(\sum_{s=1}^{m}b^p_{i_s,s}\big)^{1/p}\Big)$ ∎

Corollary 2 The equation (1.1) has the generalized solutions of arbitrary degree of M-approximation when the equation is pseudo-unsolvable. ∎

7. Estimation of degree of nearness

We take following degree of nearness as an example.

Let $A=(a_1,\ldots,a_m),D=(d_1,\ldots,d_m)\in F(V)$. Write

$$\overline{a}=\bigvee_{1\leqslant j\leqslant m}a_j \qquad \underline{a}=\bigwedge_{1\leqslant j\leqslant m}a_j$$

$$AoD=\bigvee_{1\leqslant j\leqslant m}(a_j\wedge d_j) \qquad A\bullet D=\bigwedge_{1\leqslant j\leqslant m}(a_j\vee d_j)$$

$\overline{a}$ and $\underline{a}$ is respectively called the upper norm and the lower norm of A; AoD and A●D, the interior product and the exterior product of A and D. Put

$$N(A,D)=1-(\overline{a}-\underline{a})+(AoD-A\bullet D)$$

We call N(A,D) the degree of nearness of D to A.

Theorem 7.1 Suppose $\mathcal{X}^\delta\neq\emptyset$. For any $X\in\mathcal{X}^\delta$ we have

$$N(B, XoR) \geq 1-c\delta$$

where c is a constant and $c \in [0,1]$. ∎

Corollary 1 $N(B, XoR) \geq 1-\delta$ ∎

Corollary 2 $c \leq \min_{I_k^\delta(X) \in I^\delta} \{b_{i_t, t}\}$ ∎

Corollary 3 XoR can arbitrarily be near to B when the equation (1.1) is pseudounsovable. ∎

8. Fimily for choosing and its meaning

Let $\delta^* \in [0,1]$ be a numoer chose beforehand. The equation (1.1) is called well-conditioned if the deviation of the equation $\delta^\circ \leq \delta^*$; else, ill-conditioned.

ALGORITHM 1 OF CHOOSING THE DISTRIBUTION OF THE WEIGHT NUMBER A_o

Let the equation (1.1) is well-conditioned. Choose $\delta \in (0, \delta^*]$ such that $\mathfrak{X}^\delta \neq \emptyset$. A_o can be determined in accordance with algorithm in Section3.

Definition 8.1 For any $Y=(y_1, \dots, y_n) \in \mathfrak{X}^\delta$ we put

$$A(t_1, \dots, t_n) = (y_1 + t_1(g_1 - y_1), \dots, y_n + t_n(g_n - y_n))$$

where $G=(g_1, \dots, g_n)$ is the greatest element of $\mathfrak{X}^\delta$. The following set

$$\mathcal{A} = \{ A(t_1, \dots, t_n) \mid (t_1, \dots, t_n) \in [0,1]^n \}$$

is called a fimily for choosing of the evaluation process.

ALGORITHM 2 OF CHOOSING THE DISTRIBUTION OF THE WEIGHT NUMBER A_o

First: Form the subset $\mathfrak{X}_o^\delta$ of $\mathfrak{X}^\delta$ in accordance with the algorithm in Section3, and let Y be the least element of $\mathfrak{X}_o^\delta$, then form the fimily for choosing $\mathcal{A}$.

Second: Let N(.,.) be a certain kind of degree of nearness, we make the function

$$F(t_1, \dots, t_n) = N(B, A(t_1, \dots, t_n) oR)$$

If $\exists (t_1^o, \dots, t_n^o) \in [0,1]^n$ such that

$$F(t_1^o, \dots, t_n^o) = \max_{[0,1]^n} F(t_1, \dots, t_n)$$

then we take $A_o = (t_1^o, \dots, t_n^o)$.

ALGORITHM 3 OF CHOOSING THE DISTRIBUTION OF THE WEIGHT NUMBER A_o

Let $\mathcal{Y}^\delta = \{Y_1, \dots, Y_l\}$ be the set of all the minimal elements of $\mathfrak{X}^\delta$, where $Y_i=(y_{i1},\ldots,y_{in})$. Put

$$A_i(t_1,\ldots,t_n)=(y_{i1}+t_1(g_1-y_{i1}),\ldots,y_{in}+t_n(g_n-y_{in}))$$

We get the families for choosing

$$\mathcal{A}_i=\{A_i(t_1,\ldots,t_n)\mid(t_1,\ldots,t_n)\in[0,1]^n\}\qquad i=1,\ldots,l$$

Make the functions

$$F_i(t_1,\ldots,t_n)=N(B,A_i(t_1,\ldots,t_n)\circ R)\qquad i=1,\ldots,l$$

If $\exists(t_1^i,\ldots,t_n^i)\in[0,1]^n$ such that

$$F_i(t_1^i,\ldots,t_n^i)=\max_{[0,1]^n}F_i(t_1,\ldots,t_n)$$

and let $F_s(t_1^s,\ldots,t_n^s)=\max_i\{F_i(t_1^i,\ldots,t_n^i)\}$, then we take $A_o=A(t_1^s,\ldots,t_n^s)$.

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