ON FUZZY VECTORS AND FUZZY MATRICES

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Introduction. As the title says this paper is about fuzzy vectors and fuzzy matrices. In spite of it that we have received results similar - in form - to the classical theory it will be clear to everybody after having read this paper that the presented theory is quite different from well-known fuzzy matrix theory (e.g. [3], [4], [7]). Our paper is divided into four parts. In parts 1, 2 and 3 we are concerned with the fuzzy vectors and matrices theory. The fourth part gives a more detailed analysis of solutions of fuzzy equations and of fuzzy inequalities. Material contained in this paper will be used for fuzzy programming [5].

1. The fuzzy vectors.

1.1. Let x be some element of the linear space X and $a \in \langle 0,1 \rangle$. By a fuzzy element (FE) $\{x,a\}$ it is understood a fuzzy subset of X such that $\forall y \in X$

$$\mu_{\{x,a\}}(y) = \begin{cases} a & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

If X is a set of all real numbers then $\{x,a\}$ is called the single fuzzy number (SFN).

1.11. The SFN is fuzzy number in the Dubois and Prade sense , [2], iff $a = 1$.

1.2. An m-fuzzy vector (FV) w is an ordered set of m-FEs $\{x_1,a_1\}$, $\{x_2,a_2\}, \ldots, \{x_m,a_m\}$. The FE $\{x_i,a_i\}$ is called the i-th coordinate of w. We shall use the notation $w = (\{x_i,a_i\})$, meaning w is the m-FV whose i-th coordinate is $\{x_i,a_i\}$.

1.21. An FV $w = (\{x_i,a_i\})$ such that $a_1 = a_2 = \ldots = a_m = a$ is called the fuzzy point (FP), in symbols $\{w\} = (\{x_i,a\})$.

1.3. The unit FV is the FV such that $\forall i,\ x_i = 1$. We shall denote unit FV by u or v.

1.4. The zero FV, e say, is the FV such that $\forall i,\ x_i = 0$.

Let us now define the following algebraic operations on the FVs.

1.5. Addition. If $w' = (\{x'_i,a'_i\})$ and $w'' = (\{x''_i,a''_i\})$ are m-FVs, their sum $w' + w''$ is the m-FV $(\{x'_i + x''_i, \top(a'_i,a''_i)\})$, where $\top$ is a t-norm, [6].

1.6. 1- scalar multiplication. If $w = (\{x_i,a_i\})$ is an m-FV and $\lambda \in R$, the product $\lambda \cdot w$ is the m-FV $\{\lambda \cdot x_i, a_i\}$.

1.61. We now write down some immediate and useful consequences of 1.5 and 1.6.

$(w' + w'') + w''' = w' + (w'' + w''')$ (associative law)

$w' + w'' = w'' + w'$ (commutative law)

For any $w' = (\{x_i', a_i'\})$ and $w'' = (\{x_i'', a_i''\})$ such that $a_i' \geqslant a_i''$, there is a vector w''' such that

$w' + w''' = w''$ (law of subtraction)

$\lambda \cdot (w' + w'') = \lambda \cdot w' + \lambda \cdot w''$ (vector distributive law)

$(\lambda + \mu) \cdot w = \lambda \cdot w + \mu \cdot w$ (scalar distributive law)

$\lambda \cdot (\mu \cdot w) = (\lambda \cdot \mu) \cdot w$ (scalar associative law)

$1 \cdot w = w$ (identity law)

1.7. Let $\{x,a\}$ is a FE and $\lambda \in \langle 0, 1/a \rangle$, then by the 2- scalar multiplication, $\lambda \circ \{x,a\}$ say, it is understood the FE $\{x, \lambda \cdot a\}$; we set if necessary $1/0 = \infty$.

1.8. 2- scalar multiplication. If $w = (\{x_i, a_i\})$ is an m-FV and

$\bigcap_{i=1}^{n} \langle 0, 1/a_i \rangle$, the product $\lambda \circ w$ is the FV $(\lambda \circ \{x_i, a_i\})$.

1.9. A set of FVs $w^1, w^2, \dots, w^n$ is linearly dependent, or simply dependent, if there exist numbers $b_1, \dots, b_n$, not all zero and a zero- FV e such that

$$\sum_{i=1}^{n} b_i w^i = e.$$

If the FVs are not dependent they are called independent.

1.10. An FV $w = (\{x_i, a_i\})$ is a linear combination of the FVs $w_1 = (\{x_{i1}, a_{i1}\})$, $w_2 = (\{x_{i2}, a_{i2}\}), \dots, w_n = (\{x_{in}, a_{in}\})$ if $\forall i$

$$\{x_i, a_i\} = \left(\sum_{j=1}^{n} \lambda_j \cdot \{x_{ij}, a_{ij}\} \right) \circ \beta_i$$

for some numbers λ_j and β_i.

1.11. Theorem. If each of the FVs $\bar{w}_0 = (\{y_{io}, b_{io}\}), \bar{w}_1 = (\{y_{i1}, b_{i1}\}), \dots, \bar{w}_n = (\{y_{in}, b_{in}\})$ is a linear combination of the FVs $w_1 = (\{x_{i1}, a_{i1}\}), \dots, w_n = (\{x_{in}, a_{in}\})$ then the w_i are dependent.

The proof of this theorem is similar to the crisp case.

1.12. We say that an independent set of FVs $w_1, \dots, w_n$ generates a set S of FVs iff every FV w in S is a linear combination of the w_i

1.13. The rank r of S is the maximum number of FVs which generates S

1.14. Corollary. Any set of (m+1) m-FVs is dependent.

Really, let $u^j = (\{x_{ij}, a_{ij}\})$ be an m-FV such that

$$x_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and $\forall i,j \ \ a_{ij} = 1$. Then any m-FV is a combination of the u^j, so 1.11 leads to the conclusion.

1.15. Corollary. Let $w_i = (\{x_{i1}, a_{i1}\}, \dots, \{x_{i,m+1}, a_{i,m+1}\})$ are (m+1)-FVs $(i = 1(1)m)$. Then there exist numbers $\lambda_1, \dots, \lambda_{m+1}$ not all zero and a zero-FV $e = (e^i)$ such that

$$\sum_{j=1}^{m+1} \lambda_j \cdot \{x_{ij}, a_{ij}\} = e^i \qquad i = 1(1)m.$$

Indeed, let w^j be the m-FVs befined by $w^j = (\{x_{1j}, a_{1j}\}, \dots, \{x_{mj}, a_{mj}\})$ for $j = 1(1)m+1$. Then all w^j produce a set of (m+1) m-FVs. Hence by 1.14 they are dependent. So, there exist numbers λ_j not all zero and a zero-FV, say e, such that

$$\sum_{j=1}^{m+1} \lambda_j \cdot w^j = e.$$

2. Scalar Product, Fuzzy Matrices.

Let us assume that for any $x, y \in X$, $x \cdot y \in X$.

2.1. For two m-FVs, say $w' = (\{x'_i, a'_i\})$ and $w'' = (\{x''_i, a''_i\})$, we define their product $w' \cdot w''$ to be the SFN

$$\left\{ \sum_{i=1}^{m} x'_i \cdot x''_i \ , \ \mathop{\mathrm{T}}_{i=1}^{m} (a'_i \cdot a''_i) \right\},$$

where $\mathop{\mathrm{T}}_{i=1}^{m} (a'_i \cdot a''_i) = \mathrm{T}\left(\mathop{\mathrm{T}}_{i=1}^{m-1} a'_i \cdot a''_i, \ a'_m \cdot a''_m \right).$

2.11. Some immediate properties of the scalar product are

$w' \cdot w'' = w'' \cdot w'$ (commutative law)

$(\lambda \cdot w') \cdot w'' = \lambda \cdot (w' \cdot w'')$ (mixed associative law)

2.2. An $m \times n$ fuzzy matrix (FM) is a rectangular array of FEs $\{x_{ij}, a_{ij}\}$ $(i = 1(1)m,\ j = 1(1)n)$. Thus

$$A = \begin{pmatrix} \{x_{11}, a_{11}\} & \ldots & \{x_{1n}, a_{1n}\} \\ \ldots & \ldots & \ldots \\ \{x_{m1}, a_{m1}\} & \ldots & \{x_{mn}, a_{mn}\} \end{pmatrix} .$$

Instead of writing out the above tableau, we shall simply write $A = (\{x_{ij}, a_{ij}\})$, to be read, " A is the FM whose ij-th coordinate is $\{x_{ij}, a_{ij}\}$ ".

The n-FV $w_i = (\{x_{i1}, a_{i1}\}, \ldots, \{x_{in}, a_{in}\})$ is called the i-th row FV of A.

The m-FV $w^j = (\{x_{1j}, a_{1j}\}, \ldots, \{x_{mj}, a_{mj}\})$ is called the j-th column FV of A.

The rank of the set of row FVs (column FVs) of A is called the row rank (column rank) of A.

2.3. Theorem (rank theorem). For any FM A, the row rank and column rank are equal.

The proof of this theorem is similar to the crisp case.

2.4. Corollary. If the m-FVs $w_1, \ldots, w_r$ generate the set of all m-FVs then there exist the numbers $\lambda_1, \ldots, \lambda_m$ and β such that

$$\left(\sum_{j=1}^{m} \lambda_j \cdot \{x_{ij}, a_{ij}\} \right) \circ \beta = \{y_i, b_i\} \qquad i = 1(1)r \qquad (*)$$

for any FEs $\{y_1, b_1\}, \ldots, \{y_r, b_r\}$.

Really, let A be the $(r \times m)$ FM with rows $w_1, \ldots, w_r$. By the 2.3, the columns of A have rank r, and thus if, say $z^1, \ldots, z^r$, are the set generates a set of all columns of A, we have r independent r-FVs and hence the FV $z = (\{y_i, b_i\})$ is a linear combination of them, that is

$$\{y_i, b_i\} = \left(\sum_{j=1}^{r} \lambda_j \cdot \{x_{ij}, a_{ij}\}\right) \circ \bar{\beta}$$

and hence the numbers $\lambda_1, \ldots, \lambda_r, 0, \ldots, 0, \beta$ satisfies (*), where

$$\beta = \mathop{\mathrm{T}}_{j=1}^{r} a_{ij} \cdot \bar{\beta} \,/\, \mathop{\mathrm{T}}_{j=1}^{m} a_{ij} \quad .$$

3. Fuzzy vector - fuzzy matrix product.

Let $A = (\{x_{ij}, a_{ij}\})$ be an $m \times n$ FM, let $w' = (\{y'_i, b'_i\})$ be an m-FV, and let $w'' = (\{y''_j, b''_j\})$ be an n-FV.

3.1. The product $w' \cdot A$ of w' and A is an n-FV,

$$w' \cdot A = \left(\left\{ \sum_{i=1}^{m} x_{ij} \cdot y'_i \ , \ \mathop{\mathrm{T}}_{i=1}^{m} (a_{ij} \cdot b'_i)\right\}\right).$$

3.2. The product $A \cdot w''$ of A and w'' is an m-FV

$$A \cdot w'' = \left(\left\{ \sum_{j=1}^{n} x_{ij} \cdot y''_j \ , \ \mathop{\mathrm{T}}_{j=1}^{n} (a_{ij} \cdot b''_j)\right\}\right).$$

We now write down some immediate and useful consequences of 3.1 and 3.2

$(\lambda \cdot w') \cdot A = \lambda \cdot (w' \cdot A), \quad A \cdot (\lambda \cdot w'') = \lambda \cdot (A \cdot w'')$ (homogeneity)

$(w' \cdot A) \cdot w'' = w' \cdot (A \cdot w'')$ (associativity)

The FV - FM product is a special case of the more general multiplication of two FMs.

3.3. Let $A = (\{x_{ij}, a_{ij}\})$ be an $m \times n$ FM and let $B = (\{y_{jl}, b_{jl}\})$ be an $n \times p$ FM. Then the product $A \cdot B$ of A and B is an $m \times p$ FM

$$A \cdot B = \left(\left\{ \sum_{k=1}^{n} x_{ik} \cdot y_{kj} \ , \ \mathop{\mathrm{T}}_{k=1}^{n} (a_{ik} \cdot b_{kj})\right\}\right).$$

3.4. Let $A = (\{x_{ij}, a_{ij}\})$ be an $m \times n$ FM. If λ is a number then the product $\lambda \cdot A$ is the $m \times n$ FM $(\{\lambda \cdot x_{ij}, a_{ij}\})$.

3.5. If $A=(\{x_{ij},a_{ij}\})$ is an $m\times n$ FM and λ is a number of the interval $\bigcap_{i,j}\langle 0,\ 1/a_{ij}\rangle$ then the product $\lambda\circ A$ is the $m\times n$ FM

$$\lambda\circ A=(\{x_{ij},\lambda\cdot a_{ij}\}).$$

3.6. If $A=(\{x_{ij},a_{ij}\})$ and $B=(\{y_{ij},b_{ij}\})$ are $m\times n$ FMs, then their sum $A+B$ is the $m\times n$ FM

$$A+B=(\{x_{ij}+y_{ij},\ T(a_{ij},b_{ij})\}).$$

Some properties of sum and product of FMs :

- If A is $m\times n$, B is $n\times p$, C is $p\times q$ FM , then $(A\cdot B)\cdot C=A(B\cdot C)$ (associative law)
- $\lambda\cdot(A+B)=\lambda\cdot A+\lambda\cdot B$
- $(\lambda\cdot A)\cdot B=\lambda\cdot(A\cdot B)$
- $(\lambda\circ A)\cdot B=\lambda\circ(A\cdot B)$.

4. Fuzzy equations and inequalities.

4.1. Let $\{x',a'\}$ and $\{x'',a''\}$ are the FEs. Then

- $\{x',a'\}=\{x'',a''\}$ iff $x'=x''$ and $a'=a''$,
- $\{x',a'\}<\{x'',a''\}$ or $\{x'',a''\}>\{x',a'\}$ iff $x'<x''$ and $a'\leq a''$,
- $\{x',a'\}\leq\{x'',a''\}$ or $\{x'',a''\}\geq\{x',a'\}$ iff $x'\leq x''$ and $a'\leq a''$.

4.2. Let $w'=(\{x_i',a_i'\})$ and $w''=(\{x_i'',a_i''\})$ are the FVs. Then

- $w'=w''$ iff $\forall i$ $\{x_i',a_i'\}=\{x_i'',a_i''\}$,
- $w'<w''$ iff $\forall i$ $\{x_i',a_i'\}<\{x_i'',a_i''\}$,
- $w'\leq w''$ iff $\forall i$ $\{x_i',a_i'\}\leq\{x_i'',a_i''\}$.

Let $\{w\}=(\{x_i,a\})$ be an FP $(i=1(1)n)$ and let $w'=(\{x_i',a_i\})$ be an n-FV. Because an FP is an FV, so $\{w\}\cdot w'$ is an FE

$$\{w\}\cdot w'=\{\sum_{i=1}^{n}x_i\cdot x_i'\ ,\ \operatorname*{T}_{i=1}^{n}a\cdot a_i\}. \qquad (4.1)$$

In the next part of this paper we shall consider such t-norm that

$$\operatorname*{T}_{i=1}^{n}a\cdot a_i=a\cdot\operatorname*{T}_{i=1}^{n}a_i\ .$$

4.3. From the corollary 2.4 to the rank theorem, for any n-FV $w=(\{x_i,a_i\})$ and for any FE $\{x,a\}$ there exist the numbers $\lambda_1,\dots,\lambda_n$ and β such that

$$\{x,a\}=\{\sum_{i=1}^{n}\lambda_i\cdot x_i,\ \beta\cdot\mathop{\mathrm{T}}_{i=1}^{n} a_i\} \qquad (4.2)$$

Note, that if in (4.2) $\beta\in\langle 0,1\rangle$, then (4.1) and (4.2) are equiponderant.

If $\mathop{\mathrm{T}}_{i=1}^{n} a_i \geqslant a$ then $\beta\in\langle 0,1\rangle$ and the corollary 2.4 becomes:

for any FV $w'=(\{x'_i,a'_i\})$ and for any FE $\{\bar{x},\bar{a}\}$ there exists an FP $\{w\}=(\{x_i,a\})$ such that

$$w'\cdot\{w\}=\{\bar{x},\bar{a}\} \qquad (4.3)$$

The equation (4.3) is called the fuzzy equation. The fuzzy point $\{w\}$ is called the solution of fuzzy equation (4.3).

4.4. The special case of the fuzzy equation (4.3) is the fuzzy equation

$$w'\cdot\{w\}=\{0,\bar{a}\} \qquad (4.4)$$

Note, that the set of solution of the fuzzy equation (4.4) is a fuzzy cone, [1].

4.5. Let $A=(\{x_{ij},a_{ij}\})$ be an $r\times n$ FM with the row vectors $w_1,\dots,w_r$ which generate the set of all n-FVs and let $w'=(\{x'_i,a'_i\})$ be an r-FV. If $\forall i=1(1)r$ $a'_i/\mathop{\mathrm{T}}_{j=1}^{n} a_{ij}=b\in\langle 0,1\rangle$ then the corollary 2.4 becomes:

for any $r\times n$ FM, $A=(\{x_{ij},a_{ij}\})$ say, with the row vectors, which generate the set of all n-FVs and for any r-FV $w'=(\{x'_i,a'_i\})$ such that $\forall i=1(1)r$ $a'_i/\mathop{\mathrm{T}}_{j=1}^{n} a_{ij}=b\in\langle 0,1\rangle$ there exists an FP $\{w\}=\{x_j,a\}$ such that

$$A\cdot\{w\}=w' \qquad (4.5)$$

The FP $\{w\}$ is called the solution of fuzzy equations (4.5).

4.6. Analogously, for any $r \times n$ FM, $A = (\{x_{ij}, a_{ij}\})$ say, with the column vectors which generate the set of all r-FVs and for any n-FV $w' = (\{x'_j, a'_j\})$ such that $\forall\, j = 1(1)n$

$a'_j / \mathop{\mathrm{T}}\limits_{i=1}^{r} a_{ij} = b \in \langle 0,1 \rangle$ there exists an FP $\{w\} = \{x_i, a\}$ such that

$$\{w\} \cdot A = w' \tag{4.6}$$

4.7. Let $w' = (\{x'_i, a'_i\})$ be an n-FV, $\{w\} = (\{x_i, a\})$ - an n-FP and let $\{\bar{x}, \bar{a}\}$ be an FE. The solution of fuzzy inequality

$$w' \cdot \{w\} < \{\bar{x}, \bar{a}\} \tag{4.7}$$

is the fuzzy subset P^0 such that $\forall\, y \in X^n$

$$\mu_{P^0}(y) = \begin{cases} b = \min\limits_{1 \le i \le n} \left((|\bar{a}| / \mathop{\mathrm{T}}\limits_{i=1}^{n} |a'_i| - a'_i) \vee \bar{a}/a'_i \right) \wedge 1 \\ \qquad \text{if } w' \cdot \{w\}^b < \{\bar{x}, \bar{a}\}, \\ 0 \qquad \text{otherwise,} \end{cases}$$

where $\{w\}^b = (\{y_i, b\})$.

4.8. If $\bar{P}$ is the fuzzy subset of solutions of fuzzy equation (4.3) then the fuzzy subset $P = P^0 \cup \bar{P}$ is the set of solutions of fuzzy inequality

$$w' \cdot \{w\} \le \{\bar{x}, \bar{a}\} \tag{4.8}$$

4.9. The set of solutions of fuzzy inequality

$$w' \cdot \{w\} > \{\bar{x}, \bar{a}\} \tag{4.9}$$

is the fuzzy subset P_0 such that $\forall\, y \in X^n$

$$\mu_{P_0}(y) = \begin{cases} b = \min\limits_{1 \le i \le n} (|\bar{a}| / \mathop{\mathrm{T}}\limits_{i=1}^{n} |a'_i| + a'_i) \wedge 1 \\ \qquad \text{if } w' \cdot \{w\}^b > \{\bar{x}, \bar{a}\}, \\ 0 \qquad \text{otherwise.} \end{cases}$$

4.10. Let $\{x, a\}$ be an FE. The product $(-1) * \{x, a\} = \{x, -a\}$. Analagously, if $A = (\{x_{ij}, a_{ij}\})$ is an FM and $w = (\{x_i, a_i\})$ is an FV then

- $(-1) * A = (\{x_{ij}, -a_{ij}\})$,
- $(-1) * w = (\{x_i, -a_i\})$.

Note, that the set of solutions of fuzzy inequality $w' \cdot \{w\} > \{\bar{x}, \bar{a}\}$ is equal to the set of solutions of inequality

$$(-1) * w' \cdot \{w\} \cdot (-1) < (-1) * \{\bar{x}, \bar{a}\} \cdot (-1).$$

4.11. Theorem. The sets of solutions of fuzzy inequalities $w' \cdot \{w\} \geq \{0, \bar{a}\}$ and $w' \cdot \{w\} \leq \{0, \bar{a}\}$ are the convex fuzzy cones.

4.12. Let P_i $(i = 1(1)n)$ are the fuzzy subsets of any space Y. The product $\bigcap_{i=1}^{n} P_i$ is the fuzzy subset P such that $\forall y \in Y$

$$\mu_P(y) = \min_{1 \leq i \leq n} \mu_{P_i}(y) ,$$

and the F-sum $(F) \bigcup_{i=1}^{n} P_i$ is the fuzzy subset P such that $\forall y \in Y$

$$\mu_P(y) = \begin{cases} \max\limits_{1 \leq i \leq n} \mu_{P_i}(y) & \text{if } \forall i = 1(1)n \ \mu_{P_i}(y) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider the set of inequalities

$$w_i' \cdot \{w\} \leq \{\bar{x}_i, \bar{a}_i\} \qquad i = 1(1)r \qquad (4.10)$$

This set becomes

$$A \cdot \{w\} \leq \bar{w} \qquad (4.11)$$

where A is the FM whose i-th row is w_i and $\bar{w} = (\{\bar{x}_i, \bar{a}_i\})$.

4.13. The inequalities (4.11) are called first kind inequalities iff the fuzzy subset P of solutions is equal to $\bigcap_{i=1}^{r} P_i$, where P_i is the set of solutions of inequality $w_i' \cdot \{w\} \leq \{\bar{x}_i, \bar{a}_i\}$.

4.14. The inequalities (4.11) such that $\forall i = 1(1)r$

$$\bar{a}_i / \mathop{\top}_{j=1}^{n} a_{ij} = b \in \langle 0, 1 \rangle$$ are called second kind inequalities

iff the fuzzy subset P of solutions is such that $\forall x \in X^n$

$$\mu_P(x)=\begin{cases} b & \text{if there exist at least two } k,l \text{ such that} \\ & \mu_{\bar{P}_k}(x),\mu_{\bar{P}_l}(x)\neq 0 \text{ and } \mu_{P_j}(x)\neq 0 \text{ for } j\neq k,l; \\ \min\limits_{1\leq i\leq r}\mu_{P_i}(x) & \text{otherwise.}\end{cases}$$

Now, let us consider the set of inequalities

$$w_i'\cdot\{w\}\geqslant\{\bar{x}_i,\bar{a}_i\} \qquad i=1(1)r \qquad (4.12)$$

This set becomes

$$A\cdot\{w\}\geqslant\bar{w} \qquad (4.13)$$

4.15. The inequalities (4.13) are first kind inequalities iff the fuzzy subset of solutions is equal to $(F)\bigcup\limits_{i=1}^{r}P_i$, where P_i is the set of solutions of inequality $w_i'\cdot\{w\}\geqslant\{\bar{x}_i,\bar{a}_i\}$.

4.16. The inqualities (4.13) such that $\forall i=1(1)r$ $\bar{a}_i/\prod\limits_{j=1}^{n}a_{ij}=b\in\langle 0,1\rangle$ are called second kind inequalities iff the fuzzy subset of solutions is such that $\forall x\in X^n$

$$\mu_P(x)=\begin{cases} b & \text{if there exist at least two } k,l \text{ such that} \\ & \mu_{\bar{P}_k}(x),\mu_{\bar{P}_l}(x)\neq 0 \text{ and for any } i=1(1)r\ \mu_{P_i}(x)\neq 0; \\ \max\limits_{1\leq i\leq r}\mu_{P_i}(x) & \text{if } \forall i\ \ \mu_{P_i}(x)\neq 0 \text{ and there do not} \\ & \text{exist } k,l \text{ such that } \mu_{\bar{P}_k}(x),\mu_{\bar{P}_l}(x)\neq 0; \\ 0 & \text{otherwise.}\end{cases}$$

The set of solutions of first or second kind inequalities (4.13) is equal to the set of solutions of inequalities

$$(-1)*A\cdot\{w\}\cdot(-1)\ \dot{\leqslant}\ (-1)*\bar{w}\cdot(-1),$$

where the symbol $\dot{\leqslant}$ denotes, that the set of solutions we find as in 4.17 and 4.18 respectively.

4.17. Theorem. The set of solutions of the first kind or second kind inequalities $A\cdot\{w\}\leqslant e$ is a convex fuzzy cone.

Let us consider the fuzzy inequality

$\{w\}\cdot w' \leq \{\bar{x},\bar{a}\}$,

where $\{w\}=(\{x_i,a\})$ is an m-FP, $w'=(\{x'_i,a'_i\})$ - an m-FV and $\{\bar{x},\bar{a}\}$ - an FE.

We replace the fuzzy inequality

$\{w\}\cdot w' \leq \{\bar{x},\bar{a}\}$

by the fuzzy equation

$\{\bar{w}\}\cdot\bar{w}' = \{\bar{x},\bar{a}\}$,

where $\{\bar{w}\}=(\{\bar{x}_j,a\})$ is an $(m+1)$ -FP such that $\forall j=1(1)m$ $\bar{x}_j = x_j$, $\bar{x}_{m+1}\geq 0$;
$\bar{w}'=(\{\bar{x}'_j,\bar{a}'_j\})$ is an $(m+1)$ -FV such that $\forall j=1(1)m$ $\bar{x}'_j = x'_j$,

$\bar{a}'_j = a'_j$, $\bar{x}'_{m+1}=1$ and $\bar{a}'_{m+1} = \overset{m}{\underset{j=1}{\mathrm{T}}} a'_j$.

4.18. Theorem. For any solution $\{w\}$ of the inequality $\{w\}\cdot w' \leq \{\bar{x},\bar{a}\}$ there exists exactly one solution of fuzzy equation $\{\bar{w}\}\cdot\bar{w}'=\{\bar{x},\bar{a}\}$, and for any solution $\{\bar{w}\}$ of fuzzy equation $\{\bar{w}\}\cdot\bar{w}'=\{\bar{x},\bar{a}\}$ there exists exactly one solution of $\{w\}\cdot w' \leq \{\bar{x},\bar{a}\}$.

Now, let us consider the fuzzy inequalities

$\{w\}\cdot A \leq \bar{w}$,

where $\{w\}=(\{x_i,a\})$ is an m-FP, $A=(\{x_{ij},a_{ij}\})$ - $m\times n$ FM and $\bar{w}=(\{\bar{y}_j,\bar{a}_j\})$ - an n-FV.

We replace the fuzzy inequalities

$\{w\}\cdot A \leq \bar{w}$

by the fuzzy equations

$\{\bar{w}\}\cdot\bar{A} = \bar{w}$,

where $\{\bar{w}\}=(\{\bar{x}_j,a\})$ is an $(m+n)$ FP such that $\forall j=1(1)m$ $\bar{x}_j = x_j$, $\forall j=m+1(1)m+n$ $\bar{x}_j\geq 0$; $\bar{A}=(\{\bar{x}_{ij},\bar{a}_{ij}\})$ is an $(m+n)\times n$ FM such that

$$\{\bar{x}_{ij},\bar{a}_{ij}\} = \begin{cases} \{x_{ij},a_{ij}\} & \text{if } i=1(1)m,\ j=1(1)n, \\ \{1,\bar{a}_j/b\} & \text{if } i=m+j,\ j=1(1)n, \\ \{0,1\} & \text{if } i\neq m+j,\ j=1(1)n, \end{cases}$$

where $b = \max_{1 \leq j \leq n} (\bar{a}_j / \mathop{\top}_{i=1}^{m} a_{ij})$.

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