A PROPERTY OF STRONGLY TRANSITIVE MATRICES OVER A DISTRIBUTIVE LATTICE

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Abstract

In paper[3], a property of transitive matrices over a totally ordered lattice has been proved which solves an open problem of [1]. The purpose of this note is to extend this result to a distributive lattice.

Let $\{L,\wedge,\vee, 0,1\}$ be a distributive lattice, $L^{n\times m}$ denote the set of all $n\times m$ matrices over L. D is called a transitive matrix, $D\in L^{n\times n}$, if $D^2\leq D$ holds.

DEFINITION Suppose $D=(d_{ij}) \in L^{n\times n}$. If

$$d_{ik} \leq d_{ij} \quad \text{or} \quad d_{kj} \leq d_{ij}$$

for all $i,j,k=1,2,\ldots,n$, then D is called a strongly transitive matrix.

It is obvious that strongly transitive matrices are transitive matrices over any lattice. If L is totally ordered, then strongly transitivity coincides with transitivity.

<u>THEOREM</u> For any strongly transitive matrix $D \in L^{n\times n}$, there exists a permutation matrix P such that $F=PDP^T=(f_{ij})$ satisfies $f_{ij} \nless f_{ji}$ for $i > j$.

Proof: The partial order in L is denoted as $\leqslant$, i.e.

$$a \leqslant b \iff a \wedge b = a, \ \forall a,b \in L.$$

From a theorem of Marczewski[2], any partial order in a set can be extended to a total order. Let's assume $\leqslant'$ is an extension of $\leqslant$ and $\{L, \leqslant'\}$ is a totally ordered set. L is a totally ordered distributive lattice with operation $\wedge'$ and $\vee'$

$$a \wedge' b = a \iff a \leqslant' b \iff a \vee' b = b, \ \forall a,b \in L.$$

Since D is strongly transitive, we have

$$d_{ik} \leqslant d_{ij} \quad \text{or} \quad d_{kj} \leqslant d_{ij}$$

which implies

$$d_{ik} \leqslant' d_{ij} \quad \text{or} \quad d_{kj} \leqslant' d_{ij}$$

$$d_{ik} \wedge' d_{kj} \leqslant' d_{ij}$$

$$\bigvee_{k=1,2,\cdots,n}' (d_{ik} \wedge' d_{kj}) \leqslant' d_{ij}, \quad i,j=1,2,\ldots,n.$$

The last inequalities show that D is a transitive matrix over $\{L, \wedge', \vee', 0, 1\}$. By the main result of[3], there exists a permutation matrix P such that $F = PDP^T = (f_{ij})$ satisfies

$$f_{ji} \leqslant' f_{ij} \quad \text{for } i > j.$$

Note $\leqslant'$ is a extension of $\leqslant$

$$f_{ji} \leqslant' f_{ij} \implies f_{ij} \nless f_{ji}, \ i > j,$$

and, $F = PDP^T = (f_{ij})$ over $\{L, \wedge, \vee, 0, 1\}$ is the same with that over $\{L, \wedge', \vee', 0, 1\}$. This completes the proof.

References

[1] Kim, K.H., and Roush, F.W., Generalized Fuzzy Matrices, Fuzzy Sets and Systems, 4(1980), 293---315.

[2] Marczewski, E., Fund. Math., 16(1930), 386.

[3] Peng Xiantu, BUSEFAL, 16(1983), 34---37.