A METHOD FOR ESTIMATING THE MEMBERSHIP FUNCTION OF A FUZZY SET

DINH XUÂN BA'

Department of Mathematics
Polytechnic Institute of Hanoi, Vietnam

1. INTRODUCTION

We propose a method for estimate the membership fuction of a fuzzy set starting from a matrix of experimental data (considered to a binary fuzzy relation). Professor Saaty has already show some important observations and interesting properties of this problem [4], they are useful in the case of estimation the membership function of fuzzy set.

In this paper we also introduce a problem open.
The terms and notations in this paper are used according to the author of [2] and [6].

2. PRELIMINARIES

Let U be a finite universal set (non-fuzzy) $U = \{u_1, \dots u_n\}$ and A a fuzzy sub-set of U. $\mu(u)$ is the membership function of A :

$$A = \sum_{i=1}^{n} \mu(u_i) / u_i$$

Suppose that the support of A supp A = U and the height of A were equal to 1. Form the matrix $F = \| f_{ij} \|$, where

$$f_{ij} = \frac{\mu(u_i)}{\mu(u_j)} \; ; \; i, j, = 1, \ldots, n \qquad (1)$$

which gives us :

$$f_{ij} > 0 \quad ; \quad i, j = 1, \ldots, n \qquad (2)$$

$$f_{ij} \, f_{ji} = 1 \quad ; \quad i, j = 1, \ldots, n \qquad (3)$$

$$f_{ij} \, f_{jk} = f_{ik} \; ; \; i, j, k = 1, \ldots, n \qquad (4)$$

Hence given only $f_{i, i+1}$ ($i = 1, \ldots, n - 1$) or a line(column) of F, we can determine F. If $\mu(u_r) = 1$, we have $f_{ir} = \mu(u_i)$.

In practical problems, we don't, in general, know the function $\mu(u)$. To estimate the values of this membership function, we can put u_i in comparison with u_j, thus each couple (u_i, u_j) is associated with a corresponding number f_{ij} which represents a "comparative rapport" between u_i and u_j (in other words u_i equals, according to the problem under consideration, f_{ij} sharing of u_j). The matrix F found is then considered a matrix of "experimental data" [6] (the supplement of its russian translation). If the matrix F has the properties (2), (3), (4), the function $\mu(u)$ given above is then determined solely and simply :

$$\mu(u_i) = \frac{f_{ij}}{f_{rj}} \qquad (5)$$

where $f_{rj} = \max_i f_{ij}$ and j is the index of the arbirtrary column of F.

However, the matrix of the experimental data does not in general satisfy conditions (3) and (4). For example, A is a fuzzy set of "talents" (or "capable men") in collectivity U, f_{ij} is the result of the "match" (or "competition") (u_i, u_j), therefore condition (4) has normally not taken place ($f_{12} = 0, 9$, i.e. u_1 has lost the match (u_1, u_2) and its capacity is evaluated by 0,9 of that of u_2 ; $f_{23} = 1$, i.e. u_2 is equivalent to u_3, hence it is probable that $f_{13} = 1,1$ which means u_1 is the "winner" of the match (u_1, u_3)). In this case, we have, accordingly, to look for the method for estimating the values of the membership function of the fuzzy set A. We propose below a method for estimating $\mu(u)$:

Let F be a square matrix of order n whose elements satisfy condition (2) (F is said to be a positive matrix). We establish the system :

$$\begin{cases} a_m(u_i) = \sum_{j=1}^{n} f_{ij}\, a_{m-1}(u_j) \\ a_0(u_i) = \sum_{j=1}^{n} f_{ij} \\ m = 1, 2, \ldots \quad ; \quad i = 1, \ldots, n \end{cases} \qquad (6)$$

System (6) is in the form of the model of the growth of groups of biological objects [1], where f_{ij} is the number of individuals of the type i fathered by an individual of the type j, $a_N(u_i)$ is the number of individuals of the type i farthered until the moment $t = N$, ($N = 0, 1, 2\ldots$). This problem calls for a study of the behavior of the component $a_N(u_i)$ when $N \to \infty$. Our problem seems the same if we consider f_{ij} as "the notes" of u_i obtained from u_j, it thus follows that the greater the value of N the more $a_N(u_i)$ describes precisely "the relative capacity" of u_1 within the collectivity U.

Pose :

$$A_N = (a_N(u_1), \ldots\ldots, a_N(u_n))'$$

$$N = 0, 1, 2, \ldots\ldots$$

where ' is the notation for the transposition of the matrix.

Write (6) in matricial script :

$$\begin{cases} A_m = F\, A_{m-1} \\ A_o = \left(\sum_{j=1}^{n} f_{1j}, \ldots\ldots, \sum_{j=1}^{n} f_{nj} \right)' \end{cases}$$

here F is the positive arbitrary matrix.

From which we easily get :

$$A_N = F^N A_o \qquad (7)$$

By virtue of Perron's theorem [1], there is for the positive matrix F a unique, positive, simple proper value λ whose absolute value is the greatest. Furthermore, lim A_N $(N \to \infty)$ exists uniquely and this unique limit (to a near factor) is the positive proper vector corresponding to the Perronian proper value.

Pose :

$$v = (v_1, \ldots, v_n)'$$

where :

$$v_i = v(u_i) = \lim_{N \to \infty} a_N(u_i) \qquad (8)$$

3. DEFINITION AND THEOREMS

Definition 1.

We call experimental membership function corresponding to matrix F the function $\bar{\mu}(u)$ defined by :

$$\bar{\mu}(u_i) = \frac{v(u_i)}{v_r} \tag{9}$$

where $v_r = \max \{v_1, \ldots, v_n\}$

The following theorem explains the reason for this definition.

Theorem

If positive matrix F satisfies conditions (3) and (4), the experimental membership function $\bar{\mu}(u)$ defined by (9) is then identical to the membership function $\mu(u)$ determined by (5).

Proof. Consider the characteristic equation :

$$|F - \lambda I| = \begin{vmatrix} 1-\lambda & f_{12} \cdots & f_{1n} \\ f_{21} & 1-\lambda \cdots & f_{2n} \\ \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} \quad \cdots & 1-\lambda \end{vmatrix} = 0$$

where I is the unitary matrix of order n.

To calculate the coefficients of the characteristic polynomial $|F - \lambda I|$, we have to determine all the principal minors of order $k (2 \leqslant k \leqslant n)$ of F :

$$F\begin{pmatrix} i_1 & i_2 \ldots i_k \\ i_1 & i_2 \ldots i_k \end{pmatrix} = \begin{vmatrix} f_{i_1 i_1} & f_{i_1 i_2} & \ldots & f_{i_1 i_k} \\ f_{i_2 i_1} & f_{i_2 i_2} & \ldots & f_{i_2 i_k} \\ \ldots & \ldots & \ldots & \ldots \\ f_{i_k i_1} & f_{i_k i_2} & \ldots & f_{i_k i_k} \end{vmatrix} \qquad (10)$$

$$1 \leq i_1 <..< i_2 <..< i_k \leq n$$

We have :

$$F\begin{pmatrix} i_1 & i_2 \ldots i_k \\ i_1 & i_2 \ldots i_k \end{pmatrix} = \sum_{(j_1, \ldots, j_k)} (-1)^{\gamma(j_1, \ldots, j_k)} f_{i_1 j_1} f_{i_2 j_2} \ldots f_{i_k j_k} \qquad (11)$$

here $\gamma(j_1, \ldots, j_k)$ is the number of inversions of the permutation $(j_1, \ldots, j_k)$ of $(i_1, \ldots, i_k)$.

Now we prove that for all permutations $(j_1, \ldots, j_k)$ we have :

$$f_{i_1 j_1} f_{i_2 j_2} \ldots f_{i_k j_k} = 1 \qquad (12)$$

Indeed, if there is $i_s = j_s$, then $f_{i_s j_s} = 1$ and the number of factors of the left member of (12) diminishes, consequently we can estimate that $i_s \neq j_s$ $(s = 1, \ldots, k)$ which does not violate the generality.

Because $j_1 \neq i_1$, we have $j_1 \in \{i_2, \ldots, i_k\}$. Suppose that $j_1 = i_{r_1}$ and $r_1 \in \{2, \ldots, k\}$.

In view of the fact that $j_{r_1} \neq i_{r_1}$, then $j_{r_1} \in \{i_2, \ldots, i_k\} \setminus \{i_{r_1}\}$, which follows that $j_{r_1} = i_{r_2}$ and $r_2 \in \{2, \ldots, k\} \setminus \{r_1\}$.

. . . .

As $j_{r_{k-1}} \neq i_{r_{k-1}}$ and as $\{i_2, \dots, i_k\} \setminus \{i_{r_1}, \dots; i_{r_{k-1}}\} = \emptyset$,

we thus obtain $j_{r_{k-1}} = i_1$, because $(j_1, j_{r_1}, \dots, j_{r_{k-1}})$ is a permutation of $(i_1, \dots, i_k)$.

It follows that :

$$f_{i_1 j_1} \dots f_{i_k j_k} = f_{i_1 i_{r_1}} f_{i_{r_1} i_{r_2}} \dots f_{i_{r_{k-1}} i_1} = f_{i_1 i_1} = 1$$

which proves (12)

We have therefore, at the disposition of (11) and (12) :

$$F\begin{pmatrix} i_1 & i_2 & \dots & i_k \\ i_1 & i_2 & \dots & i_k \end{pmatrix} = 0$$

For all groupings $(i_1, i_2, \dots, i_k)$ that verify (10) and $2 \leq k \leq n$

Determining the coefficients of the characteristic equation of F :

$$(-\lambda)^n + a_1(-\lambda)^{n-1} + a_2(-\lambda)^{n-2} + \dots + a_n = 0$$

We obtain :

$$a_1 = \sum_{i=1}^{n} f_{ii} = n$$

$$a_k = \sum_{(i_1, i_2, \dots, i_k)} F\begin{pmatrix} i_1 & i_2 & \dots & i_k \\ i_1 & i_2 & \dots & i_k \end{pmatrix} = 0$$

Consequently, we have the equation

$$(-\lambda)^n + n(-\lambda)^{n-1} = (-1)^n \lambda^{n-1} (\lambda - n) = 0$$

which follows that $\lambda = n$ is the Perronian proper value of F.

It is evident that the vector $v = (v_1, \dots, v_n)'$, where

$$v_i = c^- f_{im}, \quad i = 1, \dots, n,$$

Is the Peronian proper vector of F, here c is the arbitrary non-null constant and $m \in \{1, \ldots, n\}$, as $\lambda = n$ and v verify the system of the following homogeneous equation :

$$\begin{cases} (1 - \lambda)\, v_i + \sum_{j \neq i} f_{ij}\, v_j = 0 \\ i = 1, \ldots, n. \end{cases}$$

Indeed, we have :

$$(1 - n)\, c\, f_{im} + \sum_{j \neq i} c . f_{ij}\, f_{jm} =$$

$$= c\, [(1 - n)\, f_{im} + \sum_{j \neq i} f_{im}\,] =$$

$$= c\, [(1 - n)\, f_{im} + (n - 1)\, f_{im}] = 0$$

in choosing :

$$\frac{1}{c} = f_{rm} = \max_i f_{im} \quad ,$$

we get :

$$\overline{\mu}(u_i) = v_i = \frac{f_{im}}{f_{rm}} = f_{ir} = \mu(u_i)$$

The theorem is proven.

4. NUMERICAL EXAMPLES

Example 1 : Consider a trivial case :

$$F = \begin{Vmatrix} 1 \ldots\ldots 1 \\ \ldots\ldots\ldots\ldots \\ 1 \ldots\ldots 1 \end{Vmatrix}$$

Solving the system :

$$\begin{cases} (1 - n)\, v_i + \sum_{j \neq i} v_j = 0 \\ i = 1, \ldots, n \end{cases}$$

we get $\mu(u_i) = 1, \forall i$, which is the same thing : A is an universal set (non-fuzzy).

Example 2 Let $U = \{u_1, u_2, u_3\}$ and

$$F = \begin{Vmatrix} 1 & 0,4 & 0,5 \\ 2,5 & 1 & 1,25 \\ 2 & 0,8 & 1 \end{Vmatrix}$$

We have $\lambda = 3$. Solving the system :

$$\begin{cases} -2\, v_1 + 0,4\, v_2 + 0,5\, v_3 = 0 \\ 2,5\, v_1 - 2\, v_2 + 1,25\, v_3 = 0 \\ 2\, v_1 + 0,8\, v_2 - 2\, v_3 = 0 \end{cases}$$

We have $\mu(u_1) = 0,5$; $\mu(u_2) = 1$; $\mu(u_3) = 0,8$

The matrice F in examples 1 and 2 equally verify conditions (2), (3), (4). Consider examples below where F do not satisfy conditions (3) or (4).

Example 3. $U = \{u_1, u_2, u_3\}$

$$F = \begin{Vmatrix} 1 & 2 & 0,5 \\ 0,5 & 1 & 2 \\ 2 & 0,5 & 1 \end{Vmatrix}$$

We find easily that $\lambda = 3{,}5$. In solving the system :

$$\begin{cases} -\ 2{,}5\ v_1 + 2\ v_2 + 0{,}5\ v_3 = 0 \\ 0{,}5\ v_1 - 2{,}5\ v_2 + 2\ v_3 = 0 \\ 2\ v_1 + 0{,}5\ v_2 - 2{,}5\ v_3 = 0 \end{cases}$$

We obtain = $\overline{\mu}(u_1) = \overline{\mu}(u_2) = \overline{\mu}(u_3) = 1$

Example 4. $U = \{u_1, u_2, u_3\}$

$$F = \left\| \begin{matrix} 1 & 1 & 0{,}5 \\ 0{,}25 & 1 & 2 \\ 2 & 0{,}5 & 1 \end{matrix} \right\|$$

In an analogous way, we have :

$$\overline{\mu}(u_1) \approx 0{,}7 \quad ; \quad \overline{\mu}(u_2) = 1 \quad ; \quad \overline{\mu}(u_3) \approx 0{,}9$$

5. REMARKS

Remark 1. In forming the matrix of the experimental data, we should note the meaning of f_{ij} defined by formula (1) : f_{ij} represents the number that shows the measure (in the sense of the problem under consideration) of u_i when u_j is taken as an unit.

Suppose that in place of (1) we give :

$$f_{ij} = \mu(u_i) - \mu(u_j) + b \tag{13}$$

where b is the arbitrary constant provided that F be positive, then we would equally have the system (6), however, the functions $\bar{\mu}(u_i)$ determined by (9) don't in general coincide with $\mu(u_i)$ even when the following conditions (analogous to (2), (3), (4)) are fulfilled :

$$f_{ij} > 0 \qquad (2')$$

$$f_{ij} + f_{ji} = 2b \qquad (3')$$

$$f_{ij} + f_{jk} = f_{ik} + b \qquad (4')$$

Example 5. Consider a fuzzy set A of $U = \{u_1, u_2\}$ whose the membership function is given by $\mu(u_1) = 0{,}4$; $\mu(u_2) = 1$

We obtain, by virtue of (13) with b = 1 :

$$F = \left\| \begin{matrix} 1 & 0{,}4 \\ 1{,}6 & 1 \end{matrix} \right\|$$

We have :

$$\lambda = 1{,}8 \quad ; \quad v_1 = 0{,}5 \neq \mu(u_1) \quad ; \quad v_2 = 1 = \mu(u_2)$$

Remark 2. It is useful to use algorithm (6), (8), (9) to order the elements of $U = \{u_1, ..., u_n\}$ furnished by a binary fuzzy relation $f(u_i, u_j)$. Compare this algorithm to the proceeding proposed by Shimura [5] (presented in the supplement to the russian translation of [6]). This proceeding is the following :

After having calculated the values :

$$f(u_i | u_j) = \frac{f(u_i, u_j)}{\max\,[f(u_i, u_j)\ ,\ f(u_j, u_i)]} \qquad (14)$$

$$f(u_i|U) = \min_j f(u_i|u_j) \ , \qquad (15)$$

We can arrange the elements of U according to the values $f(u_i|U)$. Moreover, "the maximal element" $u_i^o \in U$ is found with the help of the equality :

$$f(u_i^o | U) = \max_{u_k \in U} f(u_k | U)$$

To compare the two methods mentioned, consider the examples below :

Example 6. $U = \{u_1, u_2, u_3\}$

$$F = \begin{Vmatrix} 0,5 & 0,5 & 0,25 \\ 0,125 & 0,5 & 1 \\ 1 & 0,25 & 0,5 \end{Vmatrix}$$

By virtue of (14) and (15), we have :

$$f(u_1 | U) = \min(\frac{0,5}{0,5} \ ; \ \frac{0,5}{0,5} \ ; \ \frac{0,25}{1}) = 0,25$$

$$f(u_2 | U) = \min(\frac{0,125}{0,5} \ ; \ \frac{0,5}{0,5} \ ; \ \frac{1}{1}) = 0,25$$

$$f(u_3 | U) = \min(\frac{1}{1} \ ; \ \frac{0,25}{1} \ ; \ \frac{0,5}{0,5}) = 0,25$$

Hence, according to proceeding [5], there is not a distinction between the elements of U.

With the aid of (6), (8), (9), we have (see example 4) :

$$\bar{\mu}(u_1) \approx 0,7 \ ; \ \bar{\mu}(u_2) = 1 \ ; \ \bar{\mu}(u_3) \approx 0,9$$

and u_2 is "maximal element" in U.

Example 7. $U = \{u_1, u_2, u_3, u_4\}$

$$F = \left\| \begin{array}{cccc} 0,5 & 0,5 & 1 & 0,25 \\ 0,5 & 0,5 & 0,25 & 0,5 \\ 0,25 & 1 & 0,5 & 1 \\ 1 & 0,5 & 0,25 & 0,5 \end{array} \right\|$$

We obtain, with the help of (14) and (15) :

$$f(u_1|U) = \min \left(\frac{0,5}{0,5} \;;\; \frac{0,5}{0,5} \;;\; \frac{1}{1} \;;\; \frac{0,25}{1}\right) = 0,25$$

$$f(u_2|U) = \min \left(\frac{0,5}{0,5} \;;\; \frac{0,5}{0,5} \;;\; \frac{0,25}{1} \;;\; \frac{0,5}{0,5}\right) = 0,25$$

$$f(u_3|U) = \min \left(\frac{0,25}{1} \;;\; \frac{1}{1} \;;\; \frac{0,5}{0,5} \;;\; \frac{1}{1}\right) = 0,25$$

$$f(u_4|U) = \min \left(\frac{1}{1} \;;\; \frac{0,5}{0,5} \;;\; \frac{0,25}{1} \;;\; \frac{0,5}{0,5}\right) = 0,25$$

However, the result of (6), (8), (9) is that the values $\bar{\mu}(u_i)$ cannot coincide (it is evident that for $\bar{\mu}(u_i) = \alpha$, $\forall i$, it should be that

$$\sum_{j=1}^{n} f_{ij} = \beta, \; \forall i).$$

<u>*Remark 3.*</u> It is probable to use algorithm (6), (8), (9) to estimate the unknown parameters $\theta_0, \ldots, \theta_k$ of the membership function $\mu(u \;;\; \theta_0, \ldots, \theta_k)$ of the fuzzy set A even when the support of A has continuum power.

For example, according to [2] certain membership functions have the form :

$$\mu(u) = (1 + a(u - u_0)^m)^{-1} \qquad (16)$$

where u_0 is known, $a > 0$ and m are the estimated parameters.

We can write (16) in the form of :

$$y = mx + \ln a \quad ,$$

here $y = \ln \dfrac{1 - \mu(u)}{\mu(u)}$, $x = \ln |u - u_0|$.

In order to estimate the parameters m and a, we have to choose a set of experiment points $U = \{u_1, \ldots\ldots, u_n\}$, form matrix F and calculate the corresponding values $\bar{\mu}(u_i)$.

To choose, in an optimal fashion, the set U, it seems advisable to use the optimal planification methods of experiments, for example the D-optimal method [3].

3. A PROBLEM

From the ideas given above, the following remark comes as a matter of course. Let $F = \|f_{ij}\|$ a square matrix with order n verifying conditions (2) and (3) (consider it as the matrix of the experimental data), $\mu(u)$ is the membership function of any fuzzy set fulfilling the conditions mentioned (supp $A = U = \{u_1, \ldots. u_n\}$ and its height is equal to 1). We designate by M the set of all these functions $\mu(u)$.

Established for $\mu \in M$ the matrix :

$$F_\mu = \|\bar{f}_{ij}\|$$

where $\bar{f}_{ij} = \dfrac{\mu(u_i)}{\mu(u_j)}$; $i, j = 1, \ldots., n$.

Consequently, F_μ fulfills conditions (2), (3), (4). Consider for the moment the matrix :

$$G_\mu = F_\mu - F = \|g_{ij}\|$$

Definition 2. We call optimal experimental membership function corresponding to F the function $\tilde{\mu} \in M$ minimizing the functional

$$S(\mu) = \sum_{i,j=1}^{n} g_{ij}^2 \quad , \mu \in M$$

or, which is the same thing

$$S(\tilde{\mu}) \leqslant S(\mu), \forall \mu \in M$$

We easily prove, in the case of n = 2 that :

$$\begin{cases} \overline{\mu}(u_i) = \tilde{\mu}(u_i) \\ i = 1, \ldots , n \end{cases} \tag{17}$$

The equalities (17) are also verified by F and U given in example 3. Isn't conclusion (17) fulfilled for all whole numbers $n \geqslant 3$. ?

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