Introduction of Extension Set

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§ 1. Preface

In the realistic world, there are many problems that incident to effect and the conditions being given are contradictory. But they can be made solvable through some methods of transformations. For example, it is impossible for one to utilize a steelyard to weigh an elephant. However, the story "Chao Chong weighs the elephant" revealed a possible way for solving such kind of problems, known as non-compatible problems in mathematical terminology.

In order to be adle to solve a non-compatible problem, we must consider the following three aspects:

1. The change of the matter and their characteristics.

When Chao Chong was going to weigh the elephant, if he only considered the relation of the amount, he wouldn't have solved this problem. But he was succeeded to turn the weight of an elephant into the same weight of stones. Stones can be weighed by parts, so this problem had been solved easily. The transformation of the weight of elephant turn into the weight of stones is the key to solve the problem. This is what we called the transformation of matter.

In order to move a machine, which is taller than the door of a workshop, we usually use the method of turning it upside down. Here neither machine nor its size is changed.

But the problem can be solved by exchanging the length and the height in time of moving. Therefore in solving a non-compatible problem, we must consider the characteristics of the matter and changes of characteristics.

2. Using certain non-mathematical methods.

Suppose we are wanted to prepare a chemical solution and are required to keep the foreign matters in water less than 0.01%. Evidenty, water with forgin matter of 1% can't be used. The problem can't be solved if we only take the amount into consideration. Laboratory experimenters usually heat water of 30°C to 100°C, to turn into steam, send it to the other container, and let it cool down to 30°C. The foreign matters will be much less. By this way, the water can be used. Here the problem lies on the fact that matter changes are resulted from the temperatures changes. As water is turndinto steam, the water and the foreign matters are separated. That is to say, in solving a non-compatible problem, we must sometimes use some non-mathematical methods including physical methods, chemical methods etc.

3. Establishing the logic, which permits some contradictory characteristic conditions.

In the classical mathematics, a true thing is true; a wrong thing is wrong. We describe a thing whether belonging to a set or not belonging to it by the numbers 0 and 1.

In the Fuzzy mathematics, [0, 1] are taken to the values of "the degree of membership". In such a way the concept of the "Fuzzy Set" can be established.

In time of solving a non-compatible problem, we must recogize that the true thing and the wrong thing can be interchanged. For example, the elephant which weights 5000kg, does not belong to the set of a thing which can be weighed by a steelyard. But it can be turned into the set by using some special methods. Therefore one study not only the pure mathematical logic, but also must study the logic which can describe contradictory characteristic.

This article establishes the concept of "extension set", in order to be able to discuss the outer element in a classical subset that can be transformed into the inner element of the subset. This is the basis for solving a non-compatible problem, by which we can establish the concept of dependent function corresponding to extension set. At the same time, logic value is extended from $\{0, 1\}$ to $(-\infty, +\infty)$. Here we use the value of dependent function to measure the relation of the element and the set. Then the qualitative description which shows a thing belonging to or not belonging to the set in the classical mathematics is extended into the quantitative description.

§ 2. Extension Set

1. Extension Set

If a subset A was given in the objects set U, an element in U which does not belong to A, must belong to $\bar{A}$ in the classical mathematics. But elements in $\bar{A}$ consists of two kinds of elements, between which there is the difference in innate character. For example, the standards of the workpiece, which was processed by a lathe, are specified with a diametter of $50^{+0.01}_{-0.01}$. Processed workpieces may be divided into two portions, standard and non-standard. But in the non-standard workpieces, there are kinds of workpieces with diameters $d < 49.99$. They are waste products. At the same time, there are also other kinds of workpieces with diameters $d > 50.01$. They may be transformed into standard by reprocess. This kind of workpieces are called "re-workable pieces". Clearly both the waste products and the "re-workable piece" products are the non-standard products. But they have difference in innate character.

In order to describe this relationship, we are establishing the concept of "extension set":

DEFINITION 2.1 : The so called "Extension subset $\tilde{X}$" in the objects set U under a restraint is indicated to provide a real number

$$K_{\tilde{X}}(u) \in (-\infty, +\infty)$$

for any $u \in U$, by which the relationship of u and $\tilde{X}$ is described. The mapping

$$K_{\tilde{X}} : U \longrightarrow (-\infty, +\infty)$$
$$u \longrightarrow K_{\tilde{X}}(u)$$

is called a dependent function of $\tilde{X}$.

$K_{\tilde{X}}(u) \geqslant 0$ shows $u \in X$ (common subset): and

$$X = \{u \mid K_{\tilde{X}}(u) \geqslant 0,\ u \in U\}$$

is called the classical field of $\tilde{X}$ (Fig 1).

$-1 \leqslant K_{\tilde{X}}(u) < 0$ shows $u \bar{\in} X$, but under that restraint u can be turned into $y \in X$; and

$$\dot{X} = \{u \mid -1 \leqslant K_{\tilde{X}}(u) < 0,\ u \in U\}$$

is called the extension field of $\tilde{X}$.

$K_{\tilde{X}}(u) < -1$ shows $u \bar{\in} X$ and it also can't be turned $y \in X$ under that restraint; and

$$\bar{\tilde{X}} = \{u \mid K_{\tilde{X}}(u) < -1,\ u \in U\}$$

is called the negative field of $\tilde{X}$.

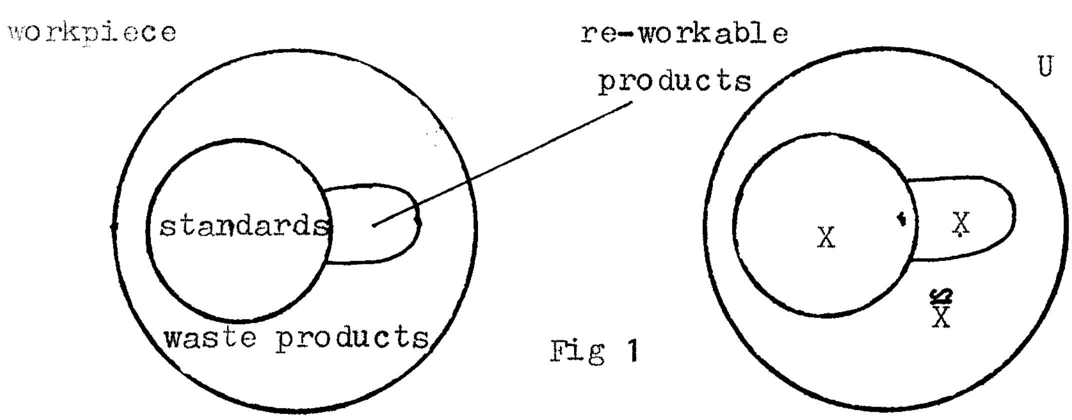

Fig 1

2. The dependent function of interval in real number field.

DEFINITION 2.2 : The dist between a point X_0 and the real interval $[a, b]$ is the number

$$d = \left| X_0 - \frac{a+b}{2} \right| - \left| \frac{b-a}{2} \right| ,$$

and written as

$$d = \rho(X_0 , X)$$

It can easily be proved that a dist has following properties:

Property 2.1 : To give a real interval ,

1) a point $x \in X$ iff $\rho(x, X) \leqslant 0$

2) a point $x \bar{\in} X$ iff $\rho(x, X) > 0$

Property 2.2 : X_1 and X_2 are two real intervals. If $X_1 \subset X_2$, then $\rho(x, X_1) \geqslant \rho(x, X_2)$ for any point $x \in (-\infty, +\infty)$.

THEOREM: If 1) X_0 and X are two intervals in real number field; 2) $X \supset X_0$; and 3) they haven't common end point; Let

$$K(y) = \frac{\rho(y, X_0)}{\rho(y, X) - \rho(y, X_0)} ,$$

then

1) $y \in X_0$ iff $K(y) \geqslant 0$

2) $y \in X - X_0$ iff $-1 \leqslant K(y) < 0$

3) $y \bar{\in} X$ iff $K(y) < -1$

Proof: (Necessity)

I. $y \in X_0$.

According to the property 2.1, we have $\rho(y, X) \leqslant 0$.

According to the property 2.2 and the given condition that X_0 and X haven't common end point, we have

$$\rho(y, X) < \rho(y, X_0)$$

$$\therefore K(y) \geqslant 0 .$$

II. $y \in X - X_0$.

According to the property 2.1, we have $\rho(y, X_0) > 0$

$\therefore K(y) \geqslant 0$.

According to the property 2.2, we have $\rho(y, X) < 0$. But

$$K(y) = \frac{\rho(y, X_0)}{\rho(y, X) - \rho(y, X_0)}$$

$$= -1 + \frac{\rho(y, X)}{\rho(y, X) - \rho(y, X_0)} ,$$

$\therefore K(y) \geqslant -1$.

III. $y \bar{\in} X$.

According to the property 2.2, we have $\rho(y, X) > 0$.

$\therefore K(y) < -1$.

(sufficiency)

I. $K(y) \geqslant 0$.

According to the property 2.2, we have

$$\rho(y, X) < \rho(y, X_0) ;$$

But

$$K(y) = \frac{\rho(y, X_0)}{\rho(y, X) - \rho(y, X_0)} \geqslant 0 ,$$

$\therefore \rho(y, X_0) \leqslant 0$.

According to the property 1.1, we have $y \in X_0$.

II. $-1 \leqslant K(y) < 0$

Since $K(y) < 0$, we have $\rho(y, X_0) > 0$; $\therefore y \bar{\in} X_0$.

According to $K(y) \geqslant -1$, we have $\rho(y, X) \leqslant 0$; $\therefore y \in X$.

And finally, we have $y \in X - X_0$.

III. $K(y) < -1$.

According to $K(y) < -1$, we have $\rho(y, X) > 0$; $\therefore y \bar{\in} X$.

Comment: When X and X_0 have common end point, we let

$$K(y) = \begin{cases} \frac{\rho(y, Xo)}{\rho(y, X) - \rho(y, X_o)} & \rho(y,X) \neq \rho(y,X_o) \\ -\rho(y, X_o) & \rho(y,X) = \rho(y,X_o) < 0 \\ -\rho(y,X_o) - 1 & \rho(y,X) = \rho(y,X_o) \geqslant 0 . \end{cases}$$

It can easily be proved that the theorem also is true for $K(y)$ above stated.

$K(y)$ which is defined by this theorem, will be a capital dependent function in this article.

3. Operations and Properties of an extension set.

Operations and properties of an extension set bear analogy to the "Fuzzy Set" [3]. Their simple definitions are as follows:

DEFINITION 2.3: If $\tilde{A}$ and $\tilde{B}$ are two extension sets in U, then the sum aggregate $\tilde{A} \cup \tilde{B}$ of $\tilde{A}$ and $\tilde{B}$, the product $\tilde{A} \cap \tilde{B}$ of $\tilde{A}$ and $\tilde{B}$, and the complement $\tilde{A}^c$ of a set $\tilde{A}$ are respectively defined by following dependent functions:

$$K_{\tilde{A} \cup \tilde{B}}(u) = \max\{K_{\tilde{A}}(u), K_{\tilde{B}}(u)\} ,$$

$$K_{\tilde{A} \cap \tilde{B}}(u) = \min\{K_{\tilde{A}}(u), K_{\tilde{B}}(u)\} ,$$

$$K_{\tilde{A}}(u) = - K_{\tilde{A}}(u) .$$

4. The immediate product of extention set.

DEFINITION 2.4: If 1) $\tilde{X}$ and $\tilde{Y}$ are respectively to the extension set of U and V; 2) their respective dependent function is $K_1(x)$ and $K_2(y)$; and 3) $W = \{w | w = xy, K_1(x) \geqslant 0, K_2(y) \geqslant 0\}$ (written as $w = X \cdot y$) [*]; then the extension set $\tilde{Z}$, which is defined by the dependent function

$$K(x, y) = \begin{cases} K_1(x) \wedge K_2(y) & xy \in X \cdot Y \\ K_1(x) \wedge K_2(y) - 1 & xy \bar{\in} X \cdot Y , \end{cases}$$

is called the immediate product of extension sets $\tilde{X}$ and $\tilde{Y}$, written as

$$\tilde{Z} = \tilde{X} \cdot \tilde{Y} \text{ .}$$

From definition 2.4, we can easily get.

Property 2.3: If 1) $\tilde{X}$ and $\tilde{Y}$ are respectively the extension set of U and V; 2) $\tilde{Z}$ is the extension set of U**X**V; and 3) $\tilde{X}=X_0 \tilde{\cup} \underset{\cdot}{X} \tilde{\cup} \tilde{\tilde{X}}$, $\tilde{y}=y_0 \tilde{\cup}\ \underset{\cdot}{y}\ \tilde{\cup}\ \tilde{\tilde{y}}$, $\tilde{Z}=Z_0 \tilde{\cup}\ \underset{\cdot}{Z}\ \tilde{\cup}\ \tilde{\tilde{Z}}$; then $\tilde{Z}=\tilde{X}\cdot\tilde{y}$ iff

$$Z_0=\{(x,y)\ |\ x \in X_0,\ y \in y_0\}$$

$$\underset{\cdot}{Z} =\{(x,y)\ |\ x \in X_0 \cup \underset{\cdot}{X},\ y \in y_0 \cup \underset{\cdot}{y},\ xy \in X \cdot y,\ (x,y) \bar{\in} Z_0\}$$

$$\tilde{\tilde{Z}} =\{(x,y)\ |\ (x,y) \in U \times V,\ (x,y) \bar{\in} Z_0 \cup \underset{\cdot}{Z}\} \text{ .}$$

Example 2.1: If 1) the range of the current intensity is U ($[0,+\infty)$; 2) the range of the voltage is V ($[0,+\infty)$; 3) X ($[a,b]$), $\underset{\cdot}{X}$ ($[0,a)$), y ($[0,c]$) and $\underset{\cdot}{y}$ ($(c,d]$) is given, where a, b, c, $d>0$, then the range of the power is $[ac,\ bd]$, and it is $X \cdot y$.

We show such extension sets through $\tilde{X}$ and $\tilde{y}$ that their classical fields are respectively X and y, and their extension fields are respectively $\underset{\cdot}{X}$ and $\underset{\cdot}{y}$; let

$$Z=\{(x,y)\ |\ a \leqslant x \leqslant b,\quad c \leqslant y \leqslant d\}$$

$$\underset{\cdot}{Z}=\{(x,y)\ |\ 0 \leqslant x \leqslant b,\ 0 \leqslant y \leqslant d,\ xy \geqslant ac,\ (x,y) \bar{\in} Z\}$$

$$\tilde{\tilde{Z}}=\{(x,y)\ |\ (x,y) \in U \times V,\ (x,y) \bar{\in} Z \cup \underset{\cdot}{Z})\} \text{ .}$$

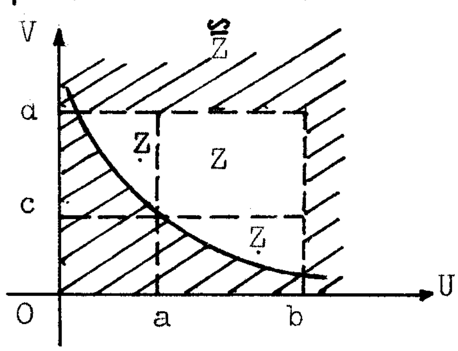

Fig 2

They form the extension set $\tilde{Z}$ (Fig 2): $\tilde{Z}= \tilde{X}\cdot\tilde{y}$.

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* $X \cdot Y=\{u\ |\ u=xy,\ x \in X,\ y \in Y\}$

Property 2.3 : If 1) $\tilde{Z} = \tilde{X} \cdot \tilde{y}$; 2) dependent functions of $\tilde{X}$, $\tilde{y}$ and $\tilde{Z}$ are respectively $K_1(x)$, $K_2(y)$ and $K(x,y)$; and 3) $K(x_o, y_o) \geqslant 0$, then $K_1(x_o) \geqslant 0$ or $K_2(y_o) \geqslant 0$ or both is true.

Prove (omit).

[1] Cai Wen Extension Set and Non-Compatible Problems SCIENCE EXPLORATION 1 (1983) CHINA.

[2] Cai Wen Essentials of Matter Element Analysis JOURNAL OF ARTIFICIAL INTELLIGENCE 2 (1983). CHINA.