EXTENSION OF FUZZY P-MEASURE

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The theorem about extension of probability measure of fuzzy events is given here. This extension bases on the notion of outer measure. Presented results are generalization of analogous results (see [2]) for crisp case. Further some new properties of fuzzy P-measure are proved.

Keywords: Fuzzy P-measure, Outer measure, Measurable fuzzy subsets, Extension of measure.

1. Introduction

In [6] probability of fuzzy events, fuzzy P-measure say, is defined as denumerable additivity measure. Its definition and some properties we can find in part 2. Proposed approach bases on the weak notions [4,7] presented in next part, too. Employment of weak notions repairs fundamental differences between fuzzy and crisp theories of probability. Therefore, the notion of fuzzy P-measure should simplify a considerations about fuzzy events.

This paper contains the next results obtained for fuzzy P-measure.

2. Preliminary notions

Let be given the crisp set Ω and the family of fuzzy subsets $\Phi = \{\mu : \Omega \to [0,1]\}$ closed under complement and union. The next parts of this paper are based on the following notions and facts.

Definition 2.1: Each fuzzy subset $\mu \in \Phi$ fulfilling the property $\mu \leqslant 1 - \mu$ is called a W-empty set. [5]

Definition 2.2: Each fuzzy subset $\mu \in \Phi$ fulfilling the property $\mu \geqslant 1 - \mu$ is called a W-universum. [5]

Definition 2.3: Each fuzzy subset $(\mu, \nu) \in \Phi^2$ such that $\mu \leqslant 1 - \nu$ are called a W-separated sets. [5]

Theorem 2.1: Any fuzzy subset $\mu \in \Phi$ is a W-empty set iff there exists $\nu \in \Phi$ such that $\mu = \nu \wedge (1 - \nu)$. [5]

Theorem 2.2: Any fuzzy subset $\mu \in \Phi$ is a W-universum iff there exists $\nu \in \Phi$ such that $\mu = \nu \vee (1 - \nu)$. [5]

Definition 2.4: If finite or infinite sequence of fuzzy subsets $\{\nu_n\}$ fulfills the next properties:

(R1) fuzzy subsets ν_n are pairwise W-separated;

(R2) the fuzzy subset $\mu \wedge (1 - \sup_n \{\nu_n\})$ is W-empty set;

(R3) $\sup_n \{\nu_n\} \leqslant \mu$

for fixed fuzzy subset μ then it is called a repartition of μ. [7]

Theorem 2.3: If the sequence $\{\mu_n\}$ fulfills the conditions (R2) and (R3) for the fuzzy subset μ then the sequence $\{\nu_n\}$ defined by identity

$$\nu_n = \begin{cases} \mu_1 & n = 1 \\ \mu_n \wedge (1 - \max\limits_{k<n} \{\mu_k\}) & n > 1 \end{cases} \qquad (2.1)$$

is a repartition of μ . [7]

Definition 2.5: Each family of mappings $\hat{\sigma} = \{\mu : \Omega \to [0,1]\}$ fulfilling the next conditions

$$0_\Omega \in \hat{\sigma} \ ,$$

$$\forall_{\mu \in \hat{\sigma}} \quad 1 - \mu \in \hat{\sigma} \ ,$$

$$\forall_{(\mu,\nu) \in \hat{\sigma}^2} \quad \mu \vee \nu \in \hat{\sigma}$$

is called a fuzzy algebra. [3]

Definition 2.6: Each fuzzy algebra σ fulfilling additionally the condition

$$\forall_{\{\mu_n\} \in \sigma^N} \quad \sup_n \{\mu_n\} \in \sigma$$

is called a fuzzy σ-algebra. [3]

Definition 2.7: If fuzzy algebra (σ-algebra) does not contain the fuzzy subset $\left[\frac{1}{2}\right]_\Omega : \Omega \to \left\{\frac{1}{2}\right\}$ then it is called soft fuzzy algebra (σ-algebra). [6,7]

Definition 2.8: Let be given the soft fuzzy σ-algebra σ . Each mapping $p: \sigma \to R^+ \cup \{0\}$ having the following properties:

(P1) for all fuzzy subsets $\mu \in \sigma$

$$p(\mu \vee (1 - \mu)) = 1 \ ,$$

(P2) if finite or infinite sequence $\{\mu_n\} \in \sigma^N$ fulfills (R1) then

$$p(\sup_n \{\mu_n\}) = \sum_n p(\mu_n)$$

is called a fuzzy P-measure on σ . [6]

Theorem 2.4: Let be given the soft fuzzy σ-algebra σ . If a mapping $p: \sigma \longrightarrow R^+ \cup \{0\}$ is a fuzzy P-measure then it fulfills the next conditions:

(P3) For all pairs of W-separated fuzzy subsets $(\mu, \nu) \in \sigma^2$ we have

$$p(\mu \vee \nu) = p(\mu) + p(\nu) \ .$$

(P4) For all fuzzy subsets $\mu \in \sigma$ we have

$$p(\mu \wedge (1 - \mu)) = 0 \ .$$

(P5) $p(1_\Omega) = 1$.

(P6) $p(0_\Omega) = 0$.

(P7) For all fuzzy subsets $\mu \in \sigma$ we have

$$p(1 - \mu) = 1 - p(\mu) \ .$$

(P8) The mapping p is a nondecreasing function.

(P9) The mapping p transforms the family σ into $[0,1]$.

(P10) For all pairs of fuzzy subsets $(\mu, \nu) \in \sigma^2$ we have

$$p(\mu \vee \nu) + p(\mu \wedge \nu) = p(\mu) + p(\nu) \ .$$

(P11) For all pairs of fuzzy subsets $(\mu, \nu) \in \sigma^2$ such that $\mu \leq \nu$ we have

$$p(\nu \wedge (1 - \mu)) = 0 \Longrightarrow p(\mu) = p(\nu) \ .$$

(P12) If the sequence $\{\nu_n\}$ is a repartition of $\mu \in \sigma$ then we have

$$p(\mu) = \sum_n p(\nu_n) \ .$$

(P13) For each sequence of fuzzy subsets $\{\mu_n\} \in \sigma^N$ we have

$$p(\sup_n \{\mu_n\}) \leq \sum_n p(\mu_n) \ .$$

(P14) Let be given any fuzzy subset $\nu \in \sigma$ we have $p(\mu \vee \nu) = p(\mu)$ for all fuzzy subsets $\mu \in \sigma$ iff $p(\nu) = 0$.

(P15) Let be given any fuzzy subset $\nu \in \sigma$. We have $p(\mu \wedge \nu) = p(\mu)$ for all fuzzy subsets $\mu \in \sigma$ iff $p(\nu) = 1$.

(P16) For each nondecreasing sequence of fuzzy subsets $\{\mu_n\} \uparrow \mu$ we have $\{p(\mu_n)\} \uparrow p(\mu)$.

(P17) For each nonincreasing sequence of fuzzy subsets $\{\mu_n\} \downarrow \mu$ we have $\{p(\mu_n)\} \downarrow p(\mu)$.

Proof: All above thesis without (P12) and (P13) are proved in [4] or [6] .

If the sequence $\{\nu_n\}$ is a repartition of μ then by (P11) and (P2) we obtain

$$p(\mu) = p(\sup_n \{\nu_n\}) = \sum_n p(\nu_n) .$$

For any sequence $\{\mu_n\}$ the sequence $\{\nu_n\}$ defined by (2.1) is a repartition of $\sup_n \{\mu_n\}$. So, from the conditions (P12) and (P8) we have

$$p(\sup_n \{\mu_n\}) = \sum_n p(\nu_n) \leq \sum_n p(\mu_n) . \blacksquare$$

3. Outer measure

Let be given the crisp set Ω and the soft fuzzy algebra $\hat{\sigma} = \{\mu : \Omega \to [0,1]\}$. The fuzzy σ-algebra $F(\Omega)$ is a family of all fuzzy subsets of Ω . Obviously we have $\hat{\sigma} \subset F(\Omega)$. Cover of fuzzy subset μ is defined as the set $C(\mu) = \{\{\mu_n\} \mid \mu \leq \sup_n \{\mu_n\}, \forall n \in N : \mu_n \in \hat{\sigma}\}$ for each $\mu \in F(\Omega)$.

Furthermore, the mapping $p: \hat{\sigma} \to [0,1]$ is a fuzzy P-measure defined on $\hat{\sigma}$. Let us consider the next notion.

Definition 3.1: The mapping $p^*: F(\Omega) \to R$ defined for each $\mu \in F(\Omega)$ as follows

$$p^*(\mu) = \inf_{C(\mu)} \{\sum_n p(\mu_n)\} \qquad (3.1)$$

is called a outer measure induced by measure p on $\hat{\sigma}$.

Theorem 3.1: The outer measure p^* induced by the measure p is a extension of p i.e. for all $\mu \in \hat{\sigma}$ we have $p^*(\mu) = p(\mu)$.

Proof: If $\mu \in \hat{\sigma}$ then $\mu \leq \mu \vee \sup_n \{0_\Omega\}$ and therefore

$$p^*(\mu) \leq p(\mu) + \sum_n p(0_\Omega) = p(\mu) .$$

On the other hand if $\mu \in \hat{\sigma}$, $\mu_n \in \hat{\sigma}$ for all $n \in N$ and $\mu \leq \sup_n \{\mu_n\}$, then, by (P8) and (P13).

$$p(\mu) \leq \sum_n p(\mu_n) ,$$

so that $p(\mu) \leq p^*(\mu)$. This proves that p^* is indeed a extension of p. ∎

From the above thesis we get immediately.

Lemma 3.1: Any outer measure induced by fuzzy P-measure satisfies the conditions (P5) and (P6) for all fuzzy subsets.

Lemma 3.2: Any outer measure induced by fuzzy P-measure fulfills the conditions (P8), (P9), (P16) and (P17) for all fuzzy subsets.

Proof: If $\mu \in F(\Omega)$, $\nu \in F(\Omega)$, $\mu \leq \nu$ and $\{\nu_n\} \in C(\nu)$ then $\{\nu_n\} \in C(\mu)$ and therefore $p^*(\mu) \leq p^*(\nu)$.

The condition (P9) follows from (P8) and the Lemma 3.1.

If $\{\nu_n\}$ is a nondecreasing sequence of fuzzy subsets, then $\sup_n\{\nu_n\}$ and $\sup_n \{p^*(\nu_n)\}$ exist and we have

$$\sup_k \{p^*(\nu_k)\} = \sup_k \{ \inf_{C(\nu_k)} \{\sum_n p(\mu_n)\}\} = \inf_{\inf_k \{C(\nu_k)\}} \{\sum_n p(\mu_n)\} =$$

$$= \inf_{C(\sup_k \{\nu_k\})} \{\sum_n p(\mu_n)\} = p^*(\sup_k \{\nu_k\}) .$$

Its proves (P16). By analogous way, as above, we obtain (P17). ∎

Lemma 3.3: Any outer measure induced by fuzzy P-measure satisfies the condition (P13) for all fuzzy subsets.

Proof: Let us suppose that μ and μ_n $(n \in N)$ are fuzzy subsets from $F(\Omega)$ such that $\mu \leq \sup_n\{\mu_n\}$. Let ε be arbitrary positive number, and choose, for each $n \in N$, a sequence $\{\mu_{n,m}\} \in C(\mu_n)$ such that

$$\sum_m p(\mu_{n,m}) \leq p^*(\mu_n) + \frac{\varepsilon}{2^n} .$$

The possibility of such a choise follows from the definition 3.1. Then, since fuzzy subsets $\mu_{n,m}$ form a sequence from $C(\mu)$ and

$$p^*(\mu) \leq \sum_n \sum_m p(\mu_{n,m}) \leq \sum_n p^*(\mu_n) + \varepsilon .$$

The arbitrariness of ε implies that

$$p^*(\mu) \leq \sum_n p^*(\mu_n) .$$ ∎

4. Measurable fuzzy subsets.

Let p^* be an outer measure induced by fuzzy P-measure on the fuzzy σ-algebra $F(\Omega)$.

Definition 4.1: A fuzzy subset $\mu \in F(\Omega)$ is p^*-measurable if, for every $\nu \in F(\Omega)$

$$p^*(\nu) = p^*(\nu \wedge \mu) + p^*(\nu \wedge (1 - \mu)) \quad .$$

A concept of this notion follows from analogous definition formulated for classical theory of denumerable additive measure by Carathéodory [1] . The class of all p^*-measurable fuzzy subsets we will be indicate by symbol $\bar{S}$. For this class we have.

Lemma 4.1: The class $\bar{S}$ contains the fuzzy subset 0_Ω .

Proof: For each $\nu \in F(\Omega)$ we get

$$p^*(\nu \wedge 0_\Omega) + p^*(\nu \wedge (1 - 0_\Omega)) = p^*(0_\Omega) + p^*(\nu \wedge 1_\Omega) = p^*(\nu)$$

so $0_\Omega \in \bar{S}$. ■

Lemma 4.2: The class $\bar{S}$ is closed under complement.

Proof: If $\mu \in \bar{S}$ then, for each $\nu \in F(\Omega)$, we have

$$p^*(\nu) = p^*(\nu \wedge \mu) + p^*(\nu \wedge (1 - \mu)) = p^*(\nu \wedge (1 - (1 - \mu))) + p^*(\nu \wedge (1 - \mu)).$$

This proves that $1 - \mu \in \bar{S}$, too. ■

Lemma 4.3: Any outer measure p^* satisfies (P4) for every p^*-measurable fuzzy subsets.

Proof: If $\mu \in \bar{S}$ then we obtain

$$p^*(\mu) = p^*(\mu \wedge \mu) + p^*(\mu \wedge (1 - \mu)) = p^*(\mu) + p^*(\mu \wedge (1 - \mu)) \quad .$$

It implies that $p^*(\mu \wedge (1 - \mu)) = 0$. ■

Lemma 4.4: The class $\bar{S}$ is closed under union.

Proof: If μ_1 and μ_2 are in $\overline{S}$ and $\nu \in F(\Omega)$, then

$$p^*(\nu) = p^*(\nu \wedge \mu_1) + p^*(\nu \wedge (1-\mu_1)) \quad \text{(i)}$$

$$p^*(\nu \wedge \mu_1) = p^*(\nu \wedge \mu_1 \wedge \mu_2) + p^*(\nu \wedge \mu_1 \wedge (1-\mu_2)) \quad \text{(ii)}$$

$$p^*(\nu \wedge (1-\mu_1)) = p^*(\nu \wedge (1-\mu_1) \wedge \mu_2) + p^*(\nu \wedge (1-\mu_1) \wedge (1-\mu_2) \quad \text{(iii)}$$

Substituting (ii) and (iii) into (i) we obtain

$$p^*(\nu) = p^*(\nu \wedge \mu_1 \wedge \mu_2) + p^*(\nu \wedge \mu_1 \wedge (1-\mu_2)) + p^*(\nu \wedge (1-\mu_1) \wedge \mu_2) + \\ + p^*(\nu \wedge (1-\mu_1) \wedge (1-\mu_2)) \quad \text{(iv)}$$

If in equation (iv) we replace ν by $\nu \wedge (\mu_1 \vee \mu_2)$, the first three terms of the right hand side remain unaltered and the last term drops out because from the conditions (P4), (P8) and (P9) we get
$0 \leq p^*(\nu \wedge (1-\mu_1) \wedge (1-\mu_2) \wedge (\mu_1 \vee \mu_2)) \leq p^*((1-(\mu_1 \vee \mu_2)) \wedge (\mu_1 \vee \mu_2)) = 0$.

We have

$$p^*(\nu \wedge (\mu_1 \vee \mu_2)) = p^*(\nu \wedge \mu_1 \wedge \mu_2) + p^*(\nu \wedge \mu_1 \wedge (1-\mu_2)) + \\ + p^*(\nu \wedge (1-\mu_1) \wedge \mu_2) \quad . \quad (4.1)$$

Since $(1-\mu_1) \wedge (1-\mu_2) = 1-(\mu_1 \vee \mu_2)$, substituting (4.1) into (iv) yields

$$p^*(\nu) = p^*(\nu \wedge (\mu_1 \vee \mu_2)) + p^*(\nu \wedge (1-(\mu_1 \vee \mu_2)))$$

which proves that $\mu_1 \vee \mu_2 \in \overline{S}$. ■

Lemma 4.5: Any outer measure p^* fulfils (P1) and (P7) for every p^*-measurable fuzzy subsets.

Proof: If $\mu \in \overline{S}$ then we have

$$1 = p^*(\mathbb{1}_\Omega) = p^*(\mathbb{1}_\Omega \wedge \mu) + p^*(\mathbb{1}_\Omega \wedge (1-\mu)) = p^*(\mu) + p^*(1-\mu) \quad .$$

Further, $\mu \vee (1-\mu) \in \overline{S}$ and

$$p^*(\mu \vee (1-\mu)) = p^*(1-(\mu \wedge (1-\mu))) = 1 - p^*(\mu \wedge (1-\mu)) = 1 \quad . ■$$

Lemma 4.6: The class $\overline{S}$ does not contain the fuzzy subset $[\![\frac{1}{2}]\!]_{\Omega}$.

Proof: Let us suppose that $[\![\frac{1}{2}]\!]_{\Omega} \in \overline{S}$. This subset is both W-empty set and W-universum. On the other side the class $\overline{S}$ is closed under union. Therefore, from the Theorems 2.1 and 2.2 together with the conditions (P1) and (P4) we get

$$1 = p^*\left([\![\tfrac{1}{2}]\!]_{\Omega}\right) = 0 \quad ,$$

so, our assumption is false. ∎

By the Lemmas 4.1, 4.2, 4.4 and 4.6 we obtain the next conclusion.

Theorem 4.1: The class $\overline{S}$ is a soft fuzzy algebra.

Lemma 4.7: If $\{\mu_n\}$ is any sequence of p^*-measurable fuzzy subsets then

$$p^*(\nu) = p(\nu \wedge (\sup_n \{\mu_n\} \vee (1 - \sup_n \{\mu_n\}))) \qquad (4.2)$$

for every $\nu \in F(\Omega)$.

Proof: Each fuzzy subset ψ_n defined as follows

$$\psi_n = \max_{k \leq n} \{\mu_n\} \qquad (4.3)$$

is p^*-measurable. Therefore we have

$$p^*(\nu) = p^*(\nu \wedge \psi_n) + p^*(\nu \wedge (1 - \psi_n)) \qquad (i)$$

for every $\nu \in F(\Omega)$. Replacing ν by $\nu \wedge (\psi_n \vee (1 - \psi_n))$ in (i) we obtain

$$p^*(\nu) \geq p^*(\nu \wedge (\sup_k \{\mu_k\} \vee (1 - \psi_n))) \geq p^*(\nu \wedge (\psi_n \vee (1 - \psi_n))) =$$

$$= p^*(\nu \wedge (\psi_n \vee (1 - \psi_n)) \wedge \psi_n) + p^*(\nu \wedge (\psi_n \vee (1 - \psi_n)) \wedge (1 - \psi_n)) =$$

$$= p^*(\nu \wedge \psi_n) + p^*(\nu \wedge (1 - \psi_n)) = p^*(\nu)$$

for each positive integer n. Therefore, by (P17) we obtain

$$p^*(\nu) = \lim_{n\to\infty} p^*(\nu \wedge (\sup_k \{\mu_k\} \vee (1 - \psi_n))) =$$
$$= p^*(\nu \wedge (\sup_k \{\mu_k\} \wedge (1 - \sup_n \{\mu_n\}))) \ . \blacksquare$$

Lemma 4.8: If $\{\mu_n\}$ is a sequence of pairwise W-separated fuzzy subsets in $\overline{S}$ then the fuzzy subset $\sup_n \{\mu_n\}$ is p^*-measurable and we have

$$p^*(\nu \wedge \sup_n \{\mu_n\}) = \sum_n p^*(\nu \wedge \mu_n) \tag{4.4}$$

for every $\nu \in F(\Omega)$.

Proof: By the identity (4.1) we obtain

$$p^*(\nu \wedge (\mu_1 \vee \mu_2)) = p^*(\nu \wedge \mu_1) + p^*(\nu \wedge \mu_2)$$

for W-separated fuzzy subsets μ_1 and μ_2 from $\overline{S}$ because

$$0 \leq p^*(\nu \wedge \mu_1 \wedge \mu_2) \leq p^*(\mu_1 \wedge (1 - \mu_1)) = 0 \ .$$

It follows by mathematical induction that

$$p^*(\nu \wedge \psi_n) = \sum_{k=1}^{n} p^*(\nu \wedge \mu_k)$$

where ψ_n is p^*-measurable fuzzy subsets defined by (4.3). We have

$$p^*(\nu) = p^*(\nu \wedge \psi_n) + p^*(\nu \wedge (1 - \psi_n)) \geq$$
$$\geq \sum_{k=1}^{n} p^*(\nu \wedge \mu_k) + p^*(\nu \wedge (1 - \sup_n \{\mu_n\})) \ .$$

Since this is true for every n, by (P13) and the Lemma 4.7 we get

$$p^*(\nu \wedge (\sup_n \{\mu_n\} \vee (1 - \sup_n \{\mu_n\}))) = p^*(\nu) \geq$$
$$\geq \sum_k p^*(\nu \wedge \mu_k) + p^*(\nu \wedge (1 - \sup_n \{\mu_n\})) \geq$$
$$\geq p^*(\nu \wedge \sup_n \{\mu_n\}) + p^*(\nu \wedge (1 - \sup_n \{\mu_n\})) \geq$$
$$\geq p^*(\nu \wedge (\sup_n \{\mu_n\} \vee (1 - \sup_n \{\mu_n\}))) .$$

It follows the identity (4.4) and $\sup_n \{\mu_n\} \in \overline{S}$. $\blacksquare$

Lemma 4.9: If $\{\mu_n\}$ is a nondecreasing sequence of p^*-measurable fuzzy subsets then the fuzzy subset $\sup_n \{\mu_n\}$ is p^*-measurable.

Proof: If the sequence $\{\mu_n\}$ satisfies the assumptions of proved Lemma, then the sequence $\{\nu_n\}$ defined by identity

$$\nu_n = \begin{cases} \mu_1 & n = 1 \\ \mu_n \wedge (1 - \mu_{n-1}) & n > 1 \end{cases} ,$$

satisfies the assumptions of the Lemma 4.8. Therefore we have

$$p^*(\nu) = p^*(\nu \wedge \sup_n \{\nu_n\}) + p^*(\nu \wedge (1 - \sup_n \{\nu_n\})) \qquad \text{(i)}$$

and, by (4.4) we get

$$p^*(\nu \wedge \max_{k \leq n} \{\nu_k\}) = p^*(\nu \wedge \mu_1) + \sum_{k=2}^{n} p^*(\nu \wedge \mu_k \wedge (1 - \mu_{k-1})) =$$

$$= p^*(\nu \wedge \mu_1) + \sum_{k=2}^{n} (p^*(\nu \wedge \mu_k) - p^*(\nu \wedge \mu_{k-1})) = p^*(\nu \wedge \mu_n)$$

for every $\nu \in F(\Omega)$ and for $n > 1$.

This along with (P16) implies that

$$p^*(\nu \wedge \sup_n \{\mu_n\}) = \lim_{n\to\infty} p^*(\nu \wedge \mu_n) = \lim_{n\to\infty} p^*(\nu \wedge \max_{k \leq n} \{\nu_k\}) =$$

$$= p^*(\nu \wedge \sup_n \{\max_{k \leq n} \{\nu_k\}\}) = p^*(\nu \wedge \sup_n \{\nu_n\}) .$$

Substituting the last result into (i), by means of (4.2) we obtain

$$p^*(\nu \wedge (\sup_n \{\mu_n\} \vee (1 - \sup_n \{\mu_n\}))) = p^*(\nu) = p^*(\nu \wedge \sup_n \{\nu_n\}) +$$

$$+ p^*(\nu \wedge (1 - \sup_n \{\nu_n\})) \geq p^*(\nu \wedge \sup_n \{\mu_n\}) + p^*(\nu \wedge (1 - \sup_n \{\mu_n\})) \geq$$

$$\geq p^*(\nu \wedge (\sup_n \{\mu_n\} \vee (1 - \sup_n \{\mu_n\}))).$$

It proves that $\sup_n \{\mu_n\} \in \overline{S}$. ■

Theorem 4.2: The class $\overline{S}$ is a soft fuzzy σ-algebra.

Proof: Let $\{\mu_n\}$ is any sequence of p^*-measurable fuzzy subsets. Then the sequence $\{\psi_n\}$, defined by (4.3), satisfies the assumptions of the Lemma 4.9. Therefore $\sup_n \{\mu_n\} = \sup_n \{\psi_n\} \in \overline{S}$. This along with the Theorem 4.1 puts on end to proof of the Theorem 4.2. ■

Theorem 4.3: The outer measure p^* is a fuzzy P-measure on $\overline{S}$.

Proof: The condition (P2) follows from the identity (4.4) for $\nu = 1_\Omega$. The Lemma 4.5 says that outer measure satisfies (P1) for all p^*-measurable subsets, the proof of the theorems is complete. ■

Let us designate by $S(\hat{\sigma})$ the smallest soft fuzzy σ-algebra containing $\hat{\sigma}$. The family $S(\hat{\sigma})$ will be called generated by $\hat{\sigma}$. We have:

Theorem 4.4: Every fuzzy subset in $S(\hat{\sigma})$ is p^*-measurable.

Proof: If $\mu \in \hat{\sigma}$, $\nu \in F(\Omega)$, and $\varepsilon > 0$, then, by the Definition 3.1, there exists a sequence $\{\mu_n\} \in C(\nu)$, such that

$$p^*(\nu) + \varepsilon \geqslant \sum_n p(\mu_n) = \sum_n (p(\mu_n \wedge \mu) + p(\mu_n \wedge (1-\mu))) \geqslant$$

$$\geqslant p^*(\nu \wedge \mu) + p^*(\nu \wedge (1-\mu)) .$$

Furthermore, there exists a sequence $\{\nu_n\} \in C(\nu)$, such that $\{\nu_n \wedge \mu\} \in C(\nu \wedge \mu)$, $\{\nu_n \wedge (1-\mu)\} \in C(\nu \wedge (1-\mu))$ and

$$p^*(\nu \wedge \mu) + p^*(\nu \wedge (1-\mu)) + \varepsilon \geqslant \sum_n p(\nu_n \wedge \mu) + \sum_n p(\nu_n \wedge (1-\mu)) =$$

$$= \sum_n p(\nu_n) \geqslant p^*(\nu) .$$

Since this is true for every ε, it follows that μ is p^*-measurable. It follows from the fact that $\overline{S}$ is a soft fuzzy σ-algebra that $S(\hat{\sigma}) \subset \bar{S}$. ■

5. The main theorem.

Can we always extend a fuzzy P-measure on $\hat{\sigma}$ to the generated soft fuzzy σ-algebra $S(\hat{\sigma})$. The answer to this question follows from the results of foregoing sections, it is formally summarized in hereafter presented theorem about uniqueness of extension.

Theorem 5.1: If p is a fuzzy P-measure on $\hat{\sigma}$, then the outer measure p^* , defined by (3.1), is unique extension of p to $S(\hat{\sigma})$, which is a fuzzy P-measure.

Proof: The extense of presented above extension is proved in parts 3 and 4.

To prove uniquenees, suppose that p_1 and p_2 are two fuzzy P-measure on $\bar{S}$ which are extensions of p on $\hat{\sigma}$, and let M be the class of all fuzzy subsets $\mu \in \bar{S}$, for which $p_1(\mu) = p_2(\mu) = p^*(\mu)$.

Obviously we have $\hat{\sigma} \subset M$. Furthermore, by the conditions (P7) and (P16) we obtain that M is a monotone class closed under complement.

Let us define a class $K(\nu)$ as a family of all fuzzy subsets μ such that $\mu \vee \nu \in M$. We have: $\mu \in K(\nu)$ iff $\nu \in K(\mu)$. Further, each class $K(\nu)$ is a monotone slass, because M is a monotone class.

If $\nu \in \hat{\sigma}$ then $\hat{\sigma} \subset K(\nu)$, it follows from the fact that if any fuzzy subset $\mu \in \hat{\sigma}$ then $\mu \vee \nu \in \hat{\sigma} \subset M$.

If $\nu \in M$, then the values $p_i(\mu \vee \nu)$ and $p_i(\nu)$ are given explicitly for every $\mu \in K(\nu)$ and for $i = 1$ or $i = 2$. Then by the conditions (P1), (P3), (P4), (P8), (P14) and (P15) we get

$$p^*(\mu \vee \nu) = p_i(\mu \vee \nu) = p_i(\mu \vee \nu) \wedge (\nu \vee (1 - \nu))) =$$
$$= p_i(\nu \wedge ((\mu \vee \nu) \wedge (1 - \nu))) = p_i(\nu \wedge (\mu \wedge (1 - \nu)) \vee (\nu \wedge (1 - \nu)) \vee$$
$$\vee (\nu \wedge (1 - \nu))) = p_i(\nu \vee (\mu \wedge (1 - \nu)) = p_i(\nu) + p_i(\mu \wedge (1 - \nu)) =$$
$$= p^*(\nu) + p_i(\mu \wedge (1 - \nu)) \quad .$$

We see that $\mu \wedge (1 - \nu) \in M$. It along with the de Morgan Law proves that the class $K(\nu)$ is closed under complement for every $\nu \in M$.

If we take into account all above results then we get $M \subset K(\nu)$ for each $\nu \in \hat{\mathfrak{S}}$. This proves that $\mu \in K(\nu)$ for all pairs $(\mu, \nu) \in M \times \hat{\mathfrak{S}}$. Therefore $\hat{\mathfrak{S}} \subset K(\nu)$ for every $\nu \in M$. Finally we obtain $M \subset K(\mu)$ for all $\mu \in M$. It implies $\mu \in K(\nu)$ for any pair $(\mu, \nu) \in M^2$. We have proved that the family M is a fuzzy algebra.

Let $\{\mu_n\}$ be any sequence of fuzzy subsets in M . The last conclusion implies that the sequence $\{\psi_n\}$, defined by (4.3), belongs to M^N . Since M is a monotony class, it follows that $\sup_n \{\mu_n\} = \sup_n \{\psi_n\} \in M$. So, the family M is a fuzzy σ-algebra. Therefore $S(\hat{\mathfrak{S}}) \subset M$. ■

6. Acknowledgments

This paper is the result of the work of the Seminar on Fuzzy and Interval Mathematics directed by Doc. dr Jerzy Albrycht.

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