A SHORT NOTICE ON BAYES' RULE AND RELATED TOPICS

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This paper deals with the Bayes' Rule for fuzzy probability spaces. The existence and properties of exhaustive collections of mutually exclusive fuzzy events are discussed. Next, the new concept of fuzzy complementary set is introduced. Finally, the properties of a certain class of fuzzy sets, denoted by M^E, are presented.

Keywords: Exhaustive collection of mutually exclusive fuzzy events, Complement of a fuzzy set, Fuzzy set, Crisp set.

0. Introduction. The notion of an exhaustive collection of mutually exclusive fuzzy events, fuzzy ECMEE - for short, is basic for the Bayes' Rule. It is proved in this paper that the only class of fuzzy sets in which there exist fuzzy ECMEEs is the class of crisp

set. It seems that in some cases modifications of fundamental definitions are needed - an intuitive example is given. Finally, an algebra of fuzzy subsets is presented.

1.0. Preliminaries. In the following the notation of A.Kaufmann, [4], is used, thus E denotes the universal set; $f_{\underset{\sim}{A}}: E \to M \subset [0,1]$ represents the membership function of a fuzzy subset $\underset{\sim}{A}$; and M^E stands for the set of all fuzzy subsets of E .

Let the triplet $(E, \underset{\sim}{\mathcal{A}}, P)$ represent a fuzzy probability space, where $E, \underset{\sim}{\mathcal{A}}$, and P denote an universal set, a fuzzy δ-algebra of E, [3], and a fuzzy measure, [3], [4], [5], respectively. By a fuzzy event we mean any fuzzy set $\underset{\sim}{A}$ such that $\underset{\sim}{A} \in \underset{\sim}{\mathcal{A}}$.

In this paper all sets denoted by $\underset{\sim}{A}$, and $\underset{\sim}{B}$, with or without subindices, are supposed to be in a given δ-algebra $\underset{\sim}{\mathcal{A}}$.

According to L.A.Zadeh, [5], we adopt two following definitions:

1.1. Two fuzzy events A and B are said to be statistically independent iff:

$$P(\underset{\sim}{A} \cdot \underset{\sim}{B}) = P(\underset{\sim}{A}) \cdot P(\underset{\sim}{B}) \quad ,$$

where $\underset{\sim}{A} \cdot \underset{\sim}{B}$ denotes an algebraic product of $\underset{\sim}{A}$ and $\underset{\sim}{B}$:

$$f_{\underset{\sim}{A} \cdot \underset{\sim}{B}}(X) = f_{\underset{\sim}{A}}(x) \cdot f_{\underset{\sim}{B}}(x) \quad .$$

1.2. The conditional probability is defined as follows:

$$P(\underset{\sim}{A} / \underset{\sim}{B}) = P(\underset{\sim}{A} \cdot \underset{\sim}{B}) / P(\underset{\sim}{B}) \quad .$$

1.3. We say that $\{\underset{\sim}{B}_i\}_{i \in I}$ is an exhaustive collection of mutually exclusive fuzzy events, which we denote by "fuzzy EMCEE", iff:

(i) $\bigcup_{i \in I} \underset{\sim}{B}_i = E$,

(ii) $\forall\, i, j \in I,\ i \neq j \Rightarrow \underset{\sim}{B}_i \cap \underset{\sim}{B}_j = \emptyset$.

In the above definition we restated the classical definition of an exhaustive collection of mutually exclusive events /crisp ECMEE/, which is basic for the formulation of the Bayes' Rule.

2.0. The Bayes' Rule for fuzzy probability spaces. First of all, should be noted, that in a fuzzy probability spaces there exist fuzzy ECMEEs, any crisp ECMEE is a simple example.

2.1. Theorem. The Bayes' Rule: Let $\{\underset{\sim}{B}_k\}_{k\in I}$ denote a fuzzy ECMEE in a fuzzy probability space $(E, \underset{\sim}{\mathfrak{K}}, P)$ and suppose that we know $P(\underset{\sim}{B}_k)$ and $P(\underset{\sim}{A}/\underset{\sim}{B}_k)$ for $k\in I$. Then:

$$P(\underset{\sim}{B}_k/\underset{\sim}{A}) = \frac{P(\underset{\sim}{B}_k)\cdot P(\underset{\sim}{A}/\underset{\sim}{B}_k)}{\sum\limits_{i\in I} P(\underset{\sim}{B}_i)\cdot P(\underset{\sim}{A}/\underset{\sim}{B}_i)}$$

Indeed, taking into account 1.1, 1.3 as well as the facts that $\{\underset{\sim}{B}_k\cdot\underset{\sim}{A}\}_{k\in I}$ is a fuzzy ECMEE in $\underset{\sim}{A}$, and that the fuzzy measure P is additive, we obtain the above formula.

3.0. An exhaustive collection of mutually exclusive fuzzy events. We are going to prove that any fuzzy ECMEE consists only of crisp events.

3.1. Lemma. Let $\{\underset{\sim}{A}_i\}_{i\in I}$ denote a fuzzy ECMEE in E. Then $\{\text{supp}(\underset{\sim}{A}_i)\}_{i\in I}$ is a crisp ECMEE, where $\text{supp}(\underset{\sim}{A}) = \{x\in E:\ f_{\underset{\sim}{A}}(x) > 0\}$.

Proff. With respect to the assumption of 3.1 and to 1.3 $\{\underset{\sim}{A}_i\}_{i\in I}$ is a fuzzy ECMEE, which means that:

(i) $\bigcup\limits_{i\in I} \underset{\sim}{A}_i = E$

(ii) $\forall\, i,j\in I\ ,\ i\neq j \Rightarrow \underset{\sim}{A}_i\cap\underset{\sim}{A}_j = \emptyset$

From (ii) it follows that:

(iii) $\forall\, i,j\in I\ ,\ i\neq j \Rightarrow \text{supp}(\underset{\sim}{A}_i)\cap\text{supp}(\underset{\sim}{A}_j) = 0$

The relation (i) yields: $\forall\ x\in E \quad \sup\limits_{i\in I} f_{\underset{\sim}{A}_i}(x) = 1$

Thus $\forall\ x\in E\ ,\ \exists\ i\in I\ :\ x\in\text{supp}(\underset{\sim}{A}_i)$

In other words $\bigcup\limits_{i\in I}\text{supp}(\underset{\sim}{A}_i)\supset E$. Keeping in mind that

$\forall\ i\in I\ ,\ \text{supp}(\underset{\sim}{A}_i)\subset E$ /ex definitione/ we obtain

(iv) $\bigcup\limits_{i\in I}\text{supp}(\underset{\sim}{A}_i) = E$.

Relations (iii) and (iv) prove the Lemma.

3.2. Theorem. Any fuzzy ECMEE is in fact a crisp ECMEE.

Proof. Let $\{\underset{\sim}{A}_i\}_{i\in I}$ be a fuzzy ECMEE. From 3.1 it follows:

(i) $\forall\, x\in E\ ,\ \exists\ i\in I\ :\ f_{\underset{\sim}{A}_i}(x) > 0$ and $\forall\ k\in I-\{i\}\ :\ f_{\underset{\sim}{A}_k}(x) = 0$

The relations 3.1 (i) and the above yield:

(ii) $\forall\ x\in E\ :\ 1 = f_E(x) = \sup_{k\in I} f_{\underset{\sim}{A}_k}(x) = f_{\underset{\sim}{A}_i}(x)$

From (i) and (ii) it follows that the membership functions of fuzzy events $\underset{\sim}{A}_i$ /for all $i\in I$/ take the values from $[0,1]$, which means that $\{\underset{\sim}{A}_i\}_{i\in I}$ is a family of crisp sets.

4.0. The class of fuzzy subsets $M^E_{\sim}$. According to 3.2 the only class of fuzzy subsets of E in which there exist fuzzy ECMEE is the class of crisp sets - 2^E. We are going to present another class of fuzzy sets, which after modifying the definition of complement of a fuzzy set, also contains fuzzy ECMEEs. But first we give an intuituve ilustration of the suggested modification.

In everyday life we encounter many fuzzy notions like beauty,right and wrong, etc. A pure mathematical notion of this kind is "being muck greater then zero". In the following we consider an example:

4.1. Let E denote the fuzzy set of real numbers which are much greater then zero, and $f_{\underset{\sim}{E}}(x)$ stand for its membership function, as evaluated, say, by Mr.X /fig.1/. Let A denote a fuzzy event: Mr.X chooses a number $x\in(200,\ 500)$, which is much greater then zero /fig.2/. In this case a fuzzy event $\overline{\underset{\sim}{A}}$ /fig.3/ seems to fit more naturally as a complement of $\underset{\sim}{A}$ then that with membership function $f(x) = 1 - f_{\underset{\sim}{A}}(x)$ /fig.4/.

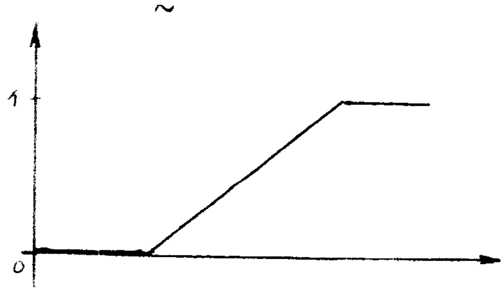

fig.1.

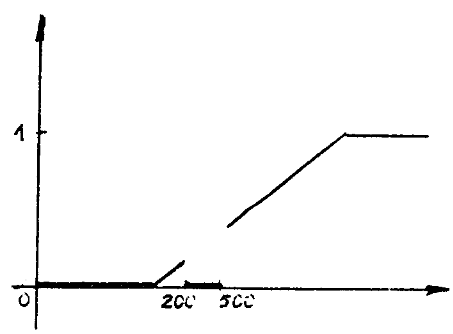

fig.3.

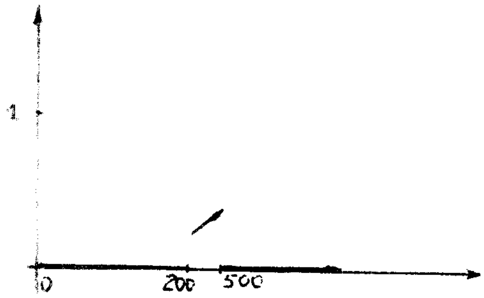

fig.2.

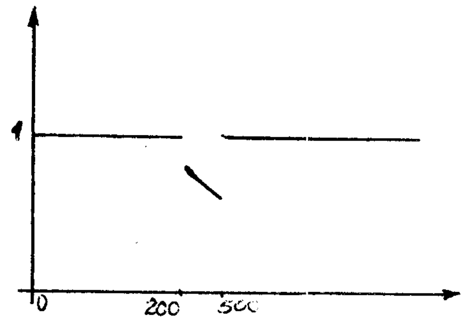

fig.4.

Now let us introduce the following definitions:

4.2. A given fuzzy subset $\underset{\sim}{E}$ of E shall be called a fuzzy universal set.

4.3. By $M^{\underset{\sim}{E}}$ we denote the class of fuzzy subsets of E such that:

$\underset{\sim}{A} \in M^{\underset{\sim}{E}}$ iff $\forall x \in E$, $f_{\underset{\sim}{A}}(x) = f_{\underset{\sim}{E}}(x)$ or $f_{\underset{\sim}{A}}(x) = 0$

4.4. Let $\underset{\sim}{A} \in M^{\underset{\sim}{E}}$. Then the fuzzy complement of $\underset{\sim}{A}$ in $M^{\underset{\sim}{E}}$ has a following membership function:

$f_{\bar{\underset{\sim}{A}}}(x) = f_{\underset{\sim}{E}}(x) - f_{\underset{\sim}{A}}(x)$

4.5. It is easily verifiable that:

$\underset{\sim}{A} \in M^{\underset{\sim}{E}}$: $\underset{\sim}{A} \cup \bar{\underset{\sim}{A}} = \underset{\sim}{E}$ and $\underset{\sim}{A} \cap \bar{\underset{\sim}{A}} = 0$

4.6. Theorem. The class $M^{\underset{\sim}{E}}$ is an algebra of fuzzy sets.

Keeping in mind 4.5, the proof of the above theorem is similar to the proof of the fact that M^{E} is merely a latice, [4] .

Let us notice that the statment 4.5 proves the existence of ECMEEs in $M^{\underset{\sim}{E}}$, which are fuzzy ex definitione. It is easily seen that the Bayes´Rule also holds in $M^{\underset{\sim}{E}}$.

5.0. Conclusion. The main purpose of this paper was to present the result formulated in 3.2 and to discuss the properties of $M^{\underset{\sim}{E}}$. We hope that the latter may find applications in decision theory.

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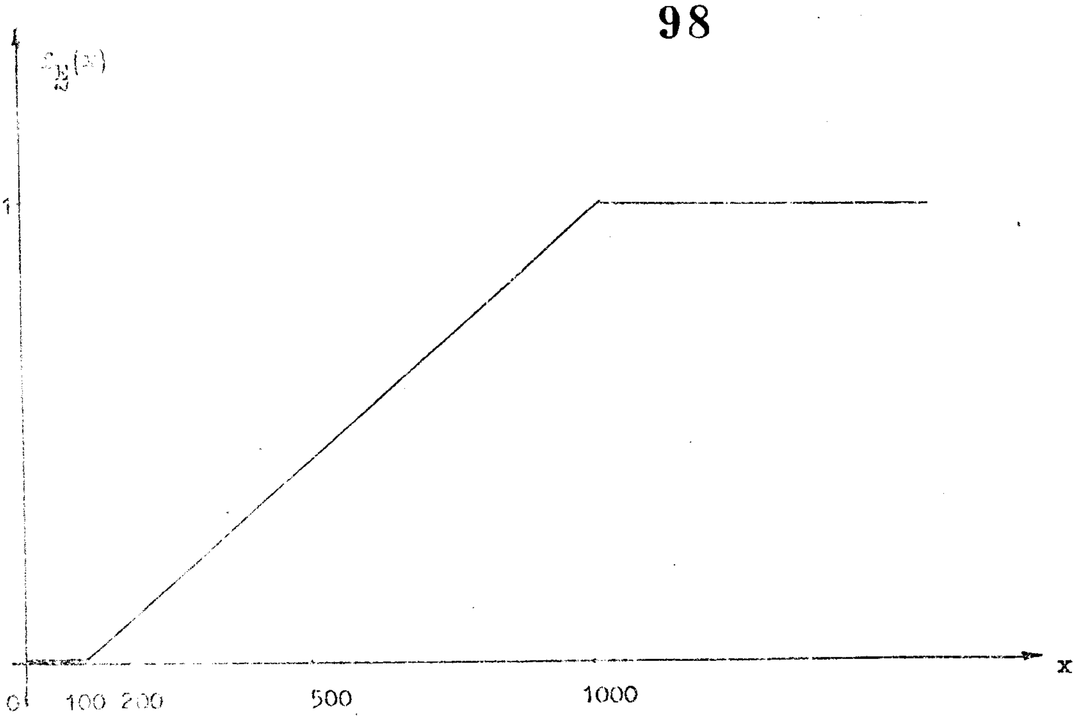


fig.1.

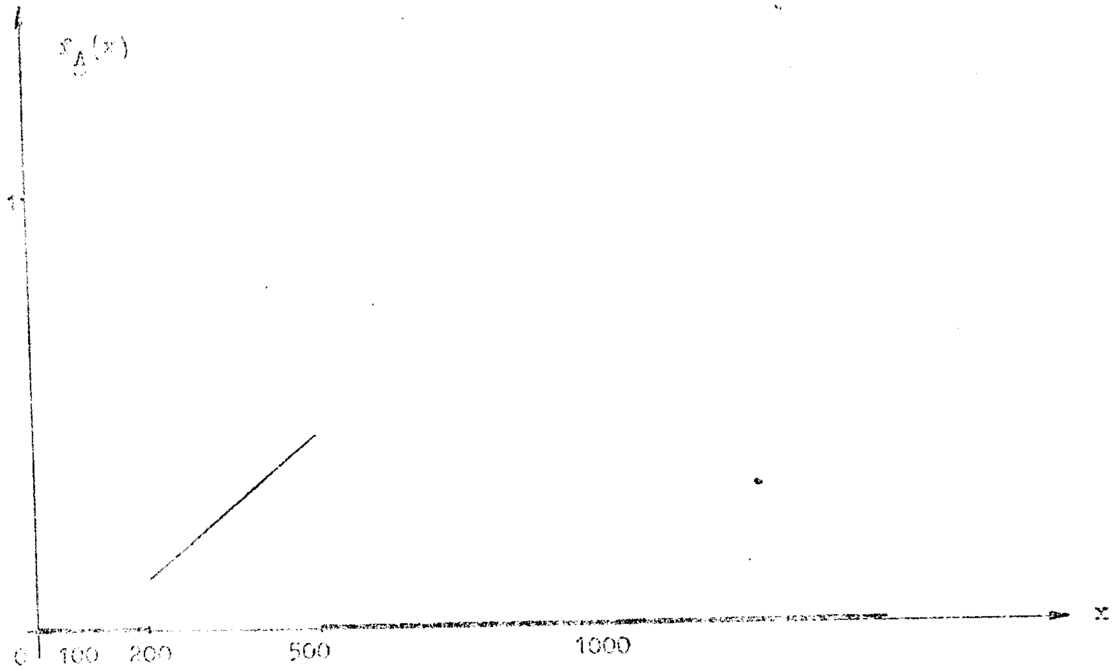
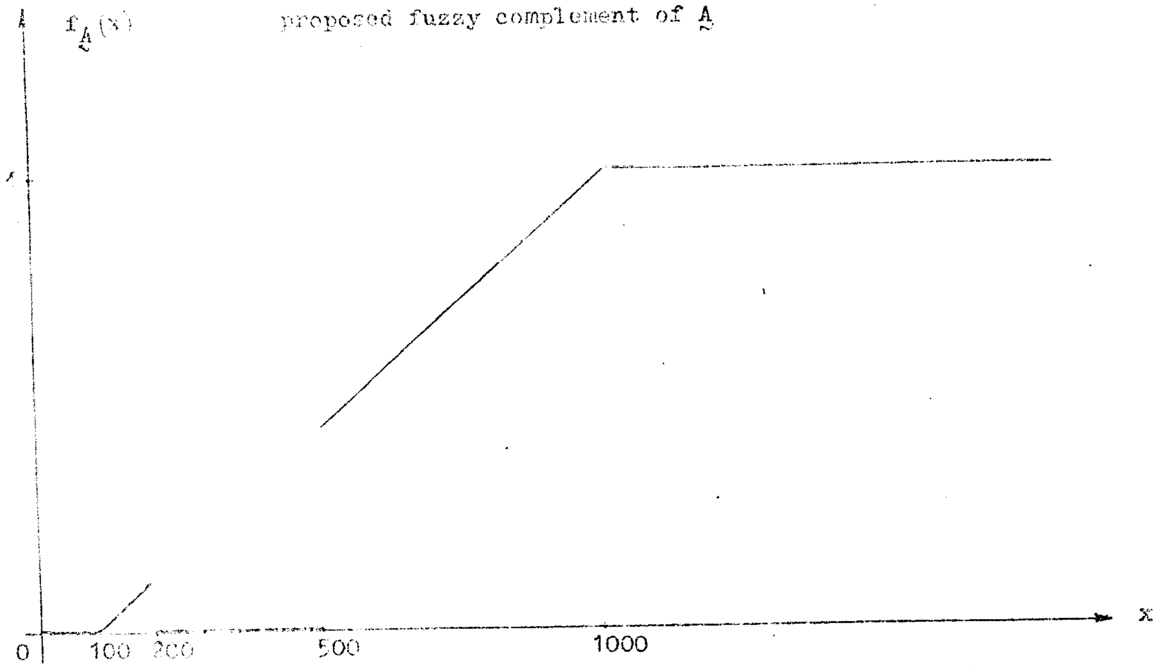
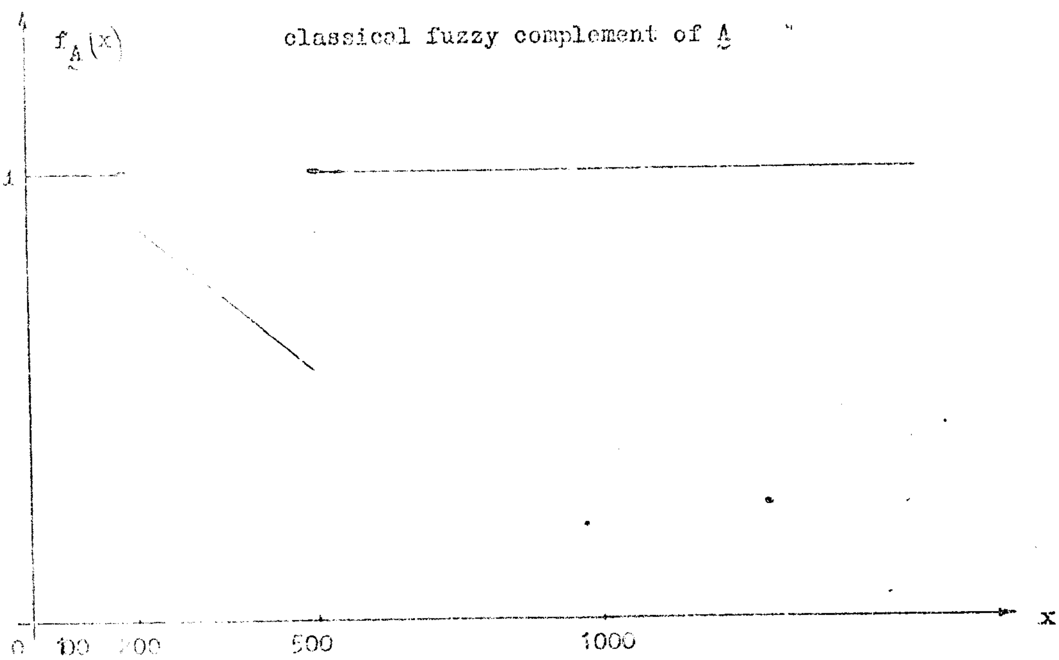


fig.2.



$f_{A}(x)$
proposed fuzzy complement of A
1
0
100
200
500
1000
x

fig.3.



$f_{A}(x)$
classical fuzzy complement of A
1
0
100
200
500
1000
x

fig.4.