HOW MANY LOWER SOLUTIONS DOES A FUZZY RELATION EQUATION HAVE?

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How many lower solutions dose a fuzzy relation equation have? E.Czagala and others have given an estimation to üpper bound in [3] . In this paper, an exact formula of number of lower solutions of the simple fuzzy relation equation will be presented.

Keywords: Fuzzy relation equations, Lower solutions

1. Introduction

We call a finite fuzzy relation equation

$$\bigvee_{i=1}^{n} (x_i \wedge a_{ij}) = b_j \quad (j = 1, \cdots, m) \tag{1.1}$$

is simple if

$$b_1 > \cdots > b_m \tag{1.2}$$

A fuzzy relation equation can be changed to a simple one holding same set of solutions whenever any two entries of b are distinct. We will give a exact formula for calculating the number of lower solutions of an arbitrary simple fuzzy relation equation.

Set

$$\overline{x}_i \triangleq \bigwedge \{ b_j \mid b_j < a_{ij} \} \quad (i = 1, \cdots, n) \tag{1.3}$$

(Promised that $\bigwedge \{ \alpha_j \mid \alpha_j \in [0,1],\ j \in \Phi \} = 1$)

$C = (c_{ij})_{n \times m}$ is called the characteristic matrix of (1.1),if

$$c_{ij} = \begin{cases} 1, & b_j \leqslant a_{ij} \leqslant \overline{x}_i \ ; \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

Set

$$L \triangleq \{ l=(i_1,\cdots,i_m) \mid 1 \leq i_j \leq n,\ c_{i_j j}>0\ (j=1,\cdots,m)\} \quad (1.5)$$

Let $\mathscr{X}$ denotes the set of solutions of (1.1),it is known that:

1. $\mathscr{X} \neq \Phi \iff L \neq \Phi$; (1.6)
2. If $\mathscr{X} \neq \Phi$,then exists a maximal solution $\bar{x}=(\bar{x}_1,\cdots,\bar{x}_n)$;
3. If $L \neq \Phi$,and for each $l=(i_1,\cdots,i_m)\in L$,let

$$\underline{x}_i(l) = \vee\{ b_j \mid i_j = i\}\ (i=1,\cdots,n) \quad (1.7)$$

(Promise that $\vee\{\alpha_j \mid \alpha_j \in [0,1],\ j\in\Phi\}=0$),then

$$\mathscr{X} = \bigcup_{l\in L} [\underline{x}(l), \bar{x}] \quad (1.8)$$

where $\underline{x}(l)=(\underline{x}_1(l),\cdots,\underline{x}_n(l))$, $[\underline{x}(l),\bar{x}]$ denotes n-dimenssional closed interval.

From (1.8) ,we can see that each lower solution belongs to $\mathscr{X}_L = \{\underline{x}(l) \mid l\in L\}$, each element $\underline{x}(l)$ of $\mathscr{X}_L$ is called a quasi-lower solution of (1.1). In order to calculate the number df lower solutions of (1.1) , we have to select a subset from $\mathscr{X}_L$,this process can be described as a pure Graph problem stating in the next.

2. Conservative ways of Boolean matrix

Given a Boolean matrix $D=(d_{ij})_{n\times m}$ $(d_{ij}\in\{0,1\})$, write

$$D_j = \{ i \mid d_{ij}=1\}\ (j=1,\cdots,m) \quad (2.1)$$

$l=(i_1,\cdots,i_m)$ is called a way of D if $i_j \in D_j$ $(j=1,\cdots,m)$.

DEFINITION 2.1 A way $l=(i_1,\cdots,i_m)$ of D is called conservative if for any $2\leq k\leq m$,if $\{i_1,\cdots,i_{k-1}\}\cap D_k \neq \Phi$, i_j is the first element encountering D_k in $(i_1,\cdots,i_{k-1})$,then holds $i_k=i_j$.

When $m=1$,every way of D is a conservative way of D.

For example, Set

$$D=\begin{pmatrix} 1&1&1&1&1&1\\ 1&0&0&0&1&1\\ 1&1&0&1&0&0\\ 1&1&0&0&0&1\\ 1&0&1&1&0&0\\ 1&1&1&0&1&0 \end{pmatrix}, \quad (2.2)$$

(1,3,5,3,2,4) is not conservative, (1,1,1,1,1,1),(3,3,1,3,1,1)... are conservative.

Given $\tau_1,\cdots,\tau_s\subset\{1,\cdots,n\}$ and $1\leq t\leq m$, ${}_{\tau_1\cdots\tau_s}^{\ t}D$ denotes a submatrix of D which has the first t colums of the matrix formed by discarding the rows occuppied by $\tau_1\cup\cdots\cup\tau_s$ from D. For example, $t=4$, $\tau_1=\{1,2,4\}$, $\tau_2=\{1,2,6\}$, then

$$_{\tau_1\tau_2\tau_3}^{\;\;4}D = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \qquad {}_{\phi}^{\,t}D = D.$$

$_{\tau_1\cdots\tau_s}^{\;\;t}(D)$ denotes the set of the all conservative ways of $_{\tau_1\cdots\tau_s}^{\;\;t}D$.

PROPOSITION 2.1

$$l = (i_1,\cdots,i_t) \in {}_{\tau}^{t}(D) \Longrightarrow l' = (i_1,\cdots,i_{t-1}) \in {}_{\tau}^{t-1}(D). \qquad (2.3)$$

PROPOSITION 2.2

$$\tau_1 \subset \tau_2 \subset \{1,2,\cdots,n\} \Longrightarrow {}_{\tau_2}^{t}(D) \subset {}_{\tau_1}^{t}(D). \qquad (2.4)$$

$|E|$ indecated the number of the elements of E .

PROPOSITION 2.3 For any $2 \leqslant t \leqslant m$ and $\tau \subset \{1,\cdots,n\}$,we have

$$|{}_{\tau}^{t}(D)| = d^*\big((|D_t\setminus\tau|-1)|{}_{\tau D_t}^{t-1}(D)| + |{}_{\tau}^{t-1}(D)|\big) \qquad (2.5)$$

where

$$d^* = \begin{cases} 1, & D_t\setminus\tau \neq \Phi; \\ 0, & D_t\setminus\tau = \Phi. \end{cases} \qquad (2.6)$$

Proof. If $d^*=0$,then $D_t\setminus\tau=\Phi$, this imples the t th column of ${}_{\tau}^{t}D$ is empty, thus ${}_{\tau}^{t}(D)=\Phi$, therfore (2.5) is ture.

Let $d^*=1$,write ${}_{\tau}^{t}(D) = M_1 \cup M_2$, where

$$M_1 = \{ l=(i_1,\cdots,i_m) \mid l \in {}_{\tau}^{t}(D),\ (i_1,\cdots,i_{t-1}) \in {}_{\tau D_t}^{t-1}(D)\}, \qquad (2.7)$$

$$M_2 = \{ l=(i_1,\cdots,i_m) \mid l \in {}_{\tau}^{t}(D),\ (i_1,\cdots,i_{t-1}) \notin {}_{\tau D_t}^{t-1}(D)\} \qquad (2.8)$$

It is easy to see $M_1 \cap M_2 = \Phi$, so that

$$|{}_{\tau}^{t}(D)| = |M_1| + |M_2|. \qquad (2.9)$$

$l \in M_1$ implies $\{i_1,\cdots,i_{t-1}\} \cap D_t = \Phi$,it may be any element in $D_t\setminus\tau$,thus

$$|M_1| = |{}_{\tau D_t}^{t-1}(D)| \cdot |D_t\setminus\tau|. \qquad (2.10)$$

From Prop. 2.1 we know, if $l=(i_1,\cdots,i_n) \in {}_{\tau}^{t}(D)$,then $(i_1,\cdots,i_{t-1}) \in {}_{\tau}^{t-1}(D)$,then $(i_1,\cdots,i_{t-1}) \in {}_{\tau}^{t-1}(D)\setminus {}_{\tau D_t}^{t-1}(D)$ .When $l \in M_2$,also $\{i_1,\cdots,i_{t-1}\} \cap D_t \neq \Phi$,from the definition of conservative ways we have that $|M_2| = |{}_{\tau}^{t-1}(D)\setminus{}_{\tau D_t}^{t-1}(D)|$, and we note from Prop. 2.2 that

$$|M_2| = |{}_{\tau}^{t-1}(D)| - |{}_{\tau D_t}^{t-1}(D)|. \qquad (2.11)$$

From (2.10) and (2.11) we know (2.5) is ture. END.

By this Prop. we can obtain the formula about $|(D)|$, for this purpose we introduce follow symbles at first:

For any $1\leqslant j\leqslant m$,let $d_j \triangleq |D_j|$;for any $1\leqslant s\leqslant m$, $1\leqslant j_1<\cdots<j_s\leqslant m$,let

$$d_{j_1\cdots j_s} \triangleq |\, D_{j_1} \setminus (D_{j_2}\cup\cdots\cup D_{j_s})\,| \tag{2.12}$$

$$d^*_{j_1\cdots j_s} \triangleq \begin{cases} 1 , & d^*_{j_1\cdots j_s} > 0 ; \\ 0 , & \text{otherwise.} \end{cases} \tag{2.13}$$

put

$$n(j_1\cdots j_s) = (d_{j_1\cdots j_s}-1)(d_{j_2\cdots j_s}-1)\cdots(d_{j_s}-1), \tag{2.14}$$

when $s=0$, we assume

$$n(j_1\cdots j_s) \triangleq n(\phi) \triangleq 1 . \tag{2.15}$$

For any $1\leqslant k\leqslant m-1$, $0\leqslant s\leqslant k$, $m-k+1\leqslant j_1<\cdots<j_s\leqslant m$,we define the sumble as follows: when $1\leqslant s\leqslant k-1$, $m-k+1<j_1<\cdots<j_s\leqslant m$,

$$\Delta^{(k)}(j_1\cdots j_s) \triangleq \{ \overline{m-k+1}\, j_1\cdots j_s , \overline{m-k+2}\, j_1\cdots j_s , \cdots, \overline{j_1-1}\, j_1\cdots j_s ; j_1 j_2\cdots j_s , \overline{j_1+1}\, j_2\cdots j_s , \overline{j_2-1}\, j_2\cdots j_s ; \cdots\cdots ; j_{s-1} j_s , \overline{j_{s-1}+1}\, j_s , \cdots, \overline{j_s-1}\, j_s ; j_s ; j_s+1 ; \cdots ; m \} \tag{2.16}$$

when $1\leqslant s\leqslant k$, $m-k+1=j_1<\cdots<j_s\leqslant m$,

$$\Delta^{(k)}(j_1\cdots j_s) \triangleq \{ j_1\cdots j_s , \overline{j_1+1}\, j_2\cdots j_s , \cdots , \overline{j_2-1}\, j_2\cdots j_s ; j_2\cdots j_s , \overline{j_2+1}\, j_3\cdots j_s , \cdots , \overline{j_3-1}\, j_3\cdots j_s ; \cdots\cdots ; j_{s-1} j_s , \overline{j_{s-1}+1}\, j_s , \cdots , \overline{j_s-1}\, j_s ; j_s ; \cdots ; m \} \tag{2.17}$$

when $s=0$

$$\Delta^{(k)}(\phi) \triangleq \{ \overline{m-k+1} ; \overline{m-k+2} ; \cdots ; m \} . \tag{2.18}$$

For example $m=12$

$$\Delta^{(10)}(579) = \{ 3579 , 4579 ; 579 , 679 ; 79 , 89 ; 9 ; 10 ; 11 ; 12 \} ;$$

$$\Delta^{(8)}(579) = \{ 579 , 679 ; 79 , 89 ; 9 ; 10 ; 11 ; 12 \} ,$$

$$\Delta^{(7)}(\Phi) = \{ 6 , 7 ; 8 , 9 , 10 , 11 , 12 \} .$$

PROPOSITION 2.4 For any $2\leqslant k\leqslant m-1$, $0\leqslant s\leqslant k-1$, $m-k+1<j_1<\cdots<j_s\leqslant m$,holds

$$\Delta^{(k)}(j_1,\cdots,j_s) = \Delta^{(k)}(\overline{m-k+1}\, j_1\cdots j_s) = \{ \overline{m-k+1}\, j_1\cdots j_s \} \cup \Delta^{(k-1)}(j_1\cdots j_s) \tag{2.19}$$

Proof. If $s=0$,then by (2.18)

$$\Delta^{(k)}(j_1\cdots j_s)=\Delta^{(k)}(\Phi)=\{\overline{m-k+1},\overline{m-k+2},\cdots,m\},$$
$$\Delta^{(k-1)}(j_1\cdots j_s)=\Delta^{(k-1)}(\Phi)=\{\overline{m-k+2},\overline{m-k+3},\cdots,m\},$$

moreover, by(2.17) we get

$$\Delta^{(k)}(\overline{m-k+1}\,j_1\cdots j_s)=\Delta^{(k)}(m-k+1)=\{\overline{m-k+1},\overline{m-k+2},\cdots,m\}.$$

which proves that (2.19) is ture.
If $s>0$, from (2.16) when $m-k+2<j_1$,we have

$$\Delta^{(k)}(j_1\cdots j_s)=\{\overline{m-k+1}\,j_1\cdots j_s\}\cup\{\overline{m-k+2}\,j_1\cdots j_s,\cdots,\overline{j_1-1}\,j_1\cdots j_s;\cdots\cdots;m\}$$
$$=\{\overline{m-k+1}\,j_1\cdots j_s\}\cup\Delta^{(k-1)}(j_1\cdots j_s);$$

when $m-k+2=j_1$,

$$\Delta^{(k)}(j_1\cdots j_s)=\{\overline{m-k+1}\,j_1\cdots j_s\}\cup\{j_1\cdots j_s,\overline{j_1+1}\,j_2\cdots j_s,\cdots,\overline{j_2-1}\,j_2\cdots j_s;\cdots\cdots;m\}$$
$$=\{\overline{m-k+1}\,j_1\cdots j_s\}\cup\Delta^{(k-1)}(j_1\cdots j_s).$$

From (2.17) we can get $\Delta^{(k)}(\overline{m-k+1}\,j_1\cdots j_s)=\Delta^{(k)}(j_1\cdots j_s)$, hence (2.19) is ture. this completes the proof of this Prop. END.

For every $1\leqslant k\leqslant m-1$, $0\leqslant s\leqslant k$, $m-k+1\leqslant j_1<\cdots<j_s\leqslant m$,write

$$\lambda^{(k)}(j_1\cdots j_s)\triangleq\prod\{d^*_{g_1\cdots g_p}\mid g_1\cdots g_p\in\Delta^{(k)}(j_1\cdots j_s)\}\qquad(2.20)$$

PROPOSITION 2.5 For any $1\leqslant k\leqslant m-1$, holds

$$|(D)|=\sum_{\substack{0\leqslant s\leqslant k\\ m-k+1\leqslant j_1<\cdots<j_s\leqslant m}}\lambda^{(k)}(j_1\cdots j_s)\,n(j_1\cdots j_s)\left|{}^{m-k}_{D_{j_1}\cdots D_{j_s}}(D)\right|\qquad(2.21)$$

Proof. With the induction for $k\in\{1,\cdots,m-1\}$, when $k=1$,in Prop. 2.3 let $t=m$, $\tau=\Phi$, we have

$$\begin{aligned}|(D)|&=d^*\left((d_m-1)\left|{}^{m-1}_{D_m}(D)\right|+\left|{}^{m-1}_{\Phi}(D)\right|\right)\\&=d^*(d_m-1)\left|{}^{m-1}_{D_m}(D)\right|+d^*\left|{}^{m-1}_{\Phi}(D)\right|\end{aligned}\qquad(2.22)$$

note that when $k=1$, $\{j_1\cdots j_s\mid 0\leqslant s\leqslant k,\ m-k+1\leqslant j_1<\cdots<j_s\leqslant m\}=\{\Phi,m\}$, we also have $\Delta^{(1)}(\Phi)=\{m\}$, $\Delta^{(1)}(m)=\{m\}$, therefore holds $\lambda^{(1)}(\Phi)=\lambda^{(1)}(m)=d^*_m=d^*$;still holds $n(\Phi)=1$, $n(m)=d_m-1$ .So (2.21) is ture.
Assuming that (2.21) be right in the case of $k-1$ ($k\leqslant m-1$),i.e.

$$|(D)| = \sum_{\substack{0 \leq s \leq K-1 \\ m-K+2 \leq j_1 < \cdots < j_s \leq m}} \lambda^{(K-1)}(j_1 \cdots j_s)\, n(j_1 \cdots j_s) \left| {}^{m-K+1}_{D_{j_1} \cdots D_{j_s}}(D) \right| . \qquad (2.23)$$

By Prop. 2.3 we have

$$\left| {}^{m-K+1}_{D_{j_1} \cdots D_{j_s}}(D) \right| = d^{*}_{\overline{m-K+1}\, j_1 \cdots j_s} \left((d_{\overline{m-K+1}\, j_1 \cdots j_s} - 1) \left| D_{m-K+1} {}^{m-K}_{D_{j_1} \cdots D_{j_s}}(D) \right| + \left| {}^{m-K}_{D_{j_1} \cdots D_{j_s}}(D) \right| \right)$$

Substitute it into (2.23), we can see that (2.21) is also ture for K . END.

THEOREM 2.1 For any Boolean matrix $D = (d_{ij})_{n \times m}$,the number of its conservative ways is

$$|(D)| = \sum_{\substack{0 \leq s \leq m-1 \\ 2 \leq j_1 < \cdots < j_s \leq m}} \lambda(j_1 \cdots j_s)\, n(j_1 \cdots j_s)\, d_{1 j_1 \cdots j_s} , \qquad (2.24)$$

where $\lambda(j_1 \cdots j_s) = \lambda^{(m-1)}(j_1 \cdots j_s)$.

Proof. In Prop. 2.5, take $K = m-1$,note that

$$\left| {}^{m-K}_{D_{j_1} \cdots D_{j_s}}(D) \right| = \left| {}^{1}_{D_{j_1} \cdots D_{j_s}}(D) \right| = \left| D_1 \setminus (D_{j_1} \cup \cdots \cup D_{j_s}) \right| = d_{1 j_1 \cdots j_s} ,$$

which can be reduced to (2.24). END.

For example, we consider the matrix D given by (2.2), according (2.24), and arrange in table for computation:

$j_1 \cdots j_s$	$d_{1 j_1 \cdots j_s}$	$\lambda(j_1 \cdots j_s)$	$d_{j_1 \cdots j_s} \cdot n(j_1 \cdots j_s)$
∅		1	6
2	4	1	6
3	3	1	6
4	3	1	6
5	3	1	6
6	3	1	6
23	2	1	2
24	2	1	2
25	2	1	2
26	2	1	2
34	1	1	0
35	1	1	0
36	2	1	2
45	2	0	2
46	2	1	2
56	1	1	0
234	1	1	0
235	1	1	0
236	1	1	0
245	1	0	0

246	1	1	0
256	1	1	0
345	0	0	0
346	1	1	0
356	1	1	0
456	2	1	0
2345	0	0	0
2346	0	0	0
2356	0	0	0
2456	0	0	0
3456	0	0	0
23456	0	0	0

At last we get $|(D)| = 48$.

3. The number of lower solutions

Given an arbitary simple fuzzy relation equation (1.2), suppose $C = (c_{ij})_{n\times m}$ be the characteristic matrix of (1.2); write

$$c_{ij}^* = \begin{cases} 1, & c_{ij} > 0 \\ 0, & c_{ij} = 0 \end{cases} \qquad (3.1)$$

and $C^* = (c_{ij}^*)$.

THEOREM 3.1 For simple fuzzy relation equation (1.2), take $D = C^*$, then the number of its lower solutions is equal to $|(D)|$.

Proof. $\mathcal{X}_L = \{\underline{x}(l) \mid l \in L\}$ is the set of the quasi-lower solutions of equation (1.2), and the set L is the set of ways of D . Choose any one $l = (i_1, \cdots, i_m) \in L$, if $l \notin (D)$, we can find the least natrual number k which satisfies that

$$2 \leqslant k \leqslant m, \quad \{i_1, \cdots, i_{k-1}\} \cap D_k \neq \Phi \ , \quad i_k \neq i_{j_0}$$

where i_{j_0} is the first element encountering D_k in $(i_1, \cdots, i_{k-1})$. Let $l' = (i_1', \cdots, i_m')$ such that $i_j' = i_j$ ($j \neq k$), $i_k' = i_{j_0}$, thus when $i \notin \{i_k, i_{j_0}\}$ we have $\underline{x}_i(l) = \underline{x}_i(l')$; and when $i = i_k$,from $i_k' = i_{j_0} \neq i_k$, we have that

$$\underline{x}_i(l') = \bigvee\{b_j \mid i_j' = i\} = \bigvee\{b_j \mid j \neq k, i_j' = i\} = \bigvee\{b_j \mid j \neq k, i_j = i\} \leqslant \bigvee\{b_j \mid i_j = i\} = \underline{x}_i(l);$$

when $i = i_{j_0}$, note that $i_k' = i_{j_0}$ and $b_{j_0} > b_k$, we have

$$\underline{x}_i(l') = \bigvee\{b_j \mid i_j' = i_{j_0}\} = \left(\bigvee\{b_j \mid j \neq k, i_j' = i_{j_0}\}\right) \bigvee \{b_k\}$$

$$= \bigvee\{b_j \mid j \neq k, i_j' = i_{j_0}\} = \bigvee\{b_j \mid j \neq k, i_j = i_{j_0}\} = \underline{x}_i(l).$$

Hence we have always $\underline{x}_i(l') \leqslant \underline{x}_i(l)$. If l' is not conservative yet, we shall improve it and the same program can be used. In the end we can find a way $l^* \in (D)$ through only finite steps, such that $\underline{x}(l^*) \leqslant \underline{x}(l)$. This shows that for any lower solution $\underline{x}$ there exists $l \in (D)$ such that $\underline{x} = \underline{x}(l)$.

Let $l = (i_1, \cdots, i_m) \in (D)$, then there is no other way $l' \in (D)$ such that $\underline{x}(l') \leqslant \underline{x}(l)$. in fact, arbitrarily take $l' = (i'_1, \cdots, i'_m) \in (D)$, assume $j_0 \leqslant \min\{j \mid i'_j \neq i_j\}$, since l and l' are both conservative ways of D , so that $\{i_1, \cdots, i_{j_0-1}\}$ $(= \{i'_1, \cdots, i'_{j_0-1}\}) \cap D_{j_0} = \Phi$ (as well as when $j_0 = 1, \{i_1, \cdots, i_{j_0-1}\} = \Phi$). Hence holds $\underline{x}_{i_{j_0}}(l') = b_{j_0} > \bigvee\{b_j \mid j > i_0\} \geqslant \bigvee\{b_j \mid i_j = i'_j\} = \underline{x}_{i'_{j_0}}(l')$, thus $\underline{x}(l') \nleqslant \underline{x}(l)$ .This asserts $\underline{x}(l)$ must be a lower solution for each $l \in (D)$ .As a matter of fact, if $l' \in L$ satisfying $\underline{x}(l') \leqslant \underline{x}(l)$,according to the discuss of the first part of the proof there exists $l'' \in (D)$ such that $\underline{x}(l'') \leqslant \underline{x}(l')$ hence $\underline{x}(l'') \leqslant \underline{x}(l)$,this contradicts the second part of this proof. The second part also shows that for each lower solution $\underline{x}$, there exists an unique way l in (D) such that $\underline{x} = \underline{x}(l)$. END.

EXAMPLE [3] gave $x \circ A = b$

$$A = \begin{pmatrix} 1 & 0.8 & 0.6 & 0.4 & 0.3 & 0.1 \\ 1 & 0.7 & 0.5 & 0.3 & 0.3 & 0.1 \\ 0.9 & 0.8 & 0.4 & 0.4 & 0.2 & 0 \\ 0.9 & 0.8 & 0.3 & 0.2 & 0.1 & 0.1 \\ 1 & 0.4 & 0.6 & 0.4 & 0 & 0 \\ 0.9 & 0.8 & 0.6 & 0 & 0.3 & 0 \end{pmatrix}$$

$$b = (0.9 \quad 0.8 \quad 0.6 \quad 0.4 \quad 0.3 \quad 0.1).$$

This is a simple equation, by (1.4) we get its characteristic matrix C, and C^* is just matrix (2.2). From Theorem 3.1 we know the number of lower solutions of it is equal to 48. E.Czogala had got the number of estimation being 106.

REFERENCE

1 E.Sanchez, Resolutions of composite fuzzy relation equation, Inform. and Contr., 30(1976)

2 A.Di Nola and S.Sessa, On the set of solutions of composite fuzzy relation equations in Brouwerian lattices, Fuzzy Sets and Systems,

3 E.Czogala, J.Drewniak, W.Pedrycz, Fuzzy relation equations on a finite set, Fuzzy Sets and Systems 7(1982) 89-101.

4 Luo Cheng-zhong, Fuzzy relation equations on finite sets, J. of Beijing Normal University, 2(1981) 37-41.