CHARACTERIZATION OF FUZZY INTERVALS *)

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Summary. Different definitions of fuzzy interval are considered and certain relations between them are presented.

1. Recent considerations of fuzzy subsets of $\mathbb{R}$. Let us consider an arbitrary fuzzy subset A of real line, i.e. function (cf. Zadeh [14])

$$A : \mathbb{R} \to [0,1].$$

Such subsets were considered in many papers as simple examples of fuzzy sets. Serious considerations of fuzzy subsets of real line are contained in the papers of Chang [2], Jain [8], Nahmias [13] and Zadeh [15]. These fuzzy sets are named fuzzy numbers or fuzzy variables without any special assumption about them. Different assumptions were made in later papers and we list here a few of them: A is a measurable, integrable function with bounded support (Mareš [9]); A is a normal convex fuzzy set (Mizumoto, Tanaka [11], [12], Dubois, Prade [6]); A is a piece-wise monotonic function with a hat-like graph (Dubois, Prade [5], Goetschel, Voxman [7]); A has exactly two intervals of monotonicity (Burdzy, Kiszka [1]). Many additional assumptions as continuity, semicontinuity, strict monotonicity, symmetry of graph or the existence of a certain limit are also used (cf. Dijkman, van Haeringen, de Lange [3]) and we can find names: fuzzy quantity, fuzzy number or fuzzy interval. From the above papers we choose a few definitions and give them a common name:"fuzzy interval".

2. Definitions of fuzzy interval. Using different assumptions on a fuzzy set A we get different notions of fuzzy interval.

Definition. By (CV), (UM) and (PM)-fuzzy interval we call a fuzzy subset A of real line with one of the following properties

(CV) $\forall_{x,y\in\mathbb{R}} \forall_{t\in(0,1)} \quad A(tx + (1-t)y) \geqslant \min(A(x),A(y))$ (convex) ,

(UM) $\exists_{a\in\mathbb{R}} \left(\forall_{x<y\leqslant a} \; A(x) \leqslant A(y), \quad \forall_{a\leqslant x<y} \; A(x) \geqslant A(y) \right)$ (uni-modal) ,

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$$(\mathrm{PM})\quad \exists_{p\leqslant q\leqslant r\leqslant s}\left(\begin{array}{l}\forall_{p\leqslant x<y\leqslant q}\ A(x)\leqslant A(y)\,,\\ \forall_{r\leqslant x<y\leqslant s}\ A(x)\geqslant A(y)\,,\\ A=0\ \text{ in }\ (-\infty,p]\cup[s,+\infty),\\ A=\sup A\ \text{ in }\ [q,r]\end{array}\right)\quad \text{(piece-wise monotonicity)},$$

respectively. Using above conditions with the strict inequalities between values of A we add the letter S before introduced symbols, e.g. we have

$$(\mathrm{SCV})\quad \forall_{x,y\in R,\ x\neq y}\ \forall_{t\in(0,1)}\ A(tx+(1-t)y) > \min(A(x),A(y))\,.$$

We also use symbols (LC) and (UC) if A is a lower or an upper semicontinuous one and we write e.g. (UC,SUM)-interval if A is an upper semicontinuous one and fulfils the strict uni-modality property.

Six introduced notions of fuzzy interval are not equivalent but we see certain interesting relations between them. At first we consider in detail the definition of (CV)-interval.

3. <u>Characterization of (CV)-interval</u>. Let us observe that the condition (CV) is equivalent to

$$(1)\quad (x<z<y \Rightarrow A(z)\geqslant \min(A(x),A(y)))\quad \text{for any}\ \ x,\,y,\,z\in\mathbb{R}\,.$$

<u>Lemma.</u> Property (CV) is equivalent to

$$(2)\qquad A(z)\geqslant \min(A_-(z),A_+(z))\quad \text{for any}\ \ z\in\mathbb{R}\,,$$

where

$$(3)\qquad A_-(z)=\sup_{x<z} A(x)\ ,\quad A_+(z)=\sup_{y>z} A(y)\ .$$

<u>Proof</u>. Let $z\in\mathbb{R}$. If A is a (CV)-interval then from (1) we get

$$A(z)\geqslant \sup_{x<z}\ \sup_{y>z}\min(A(x),A(y)) = \min(\sup_{x<z}A(x),\ \sup_{y>z}A(y))$$
$$=\min(A_-(z),A_+(z))\ .$$

So (CV) implies (3). Conversely, (1) is a direst consequence of (3) which finishes the proof.

The operation $A^{\circ}=\min(A_-,A_+)$ from (2) has the meaning of interior of fuzzy interval (consider e.g. crisp intervals). For any A, A° is a lower semicontinuous one and $(A^{\circ})^{\circ}=A^{\circ}$. In particular, if A is a (CV) or an (UM)-interval, then A° is (LC,CV) or (LC,UM)-interval, respectively. Moreover, for any (LC,CV), (LC,UM) or (LC,PM)-interval we have $A=A^{\circ}$.

From [4] § 21 we rewrite the two following theorems:

Theorem 1. $A \neq 0_{\mathbb{R}}$ is a (CV)-interval iff there exist $p, q, r, s \in \overline{\mathbb{R}}$ such that

(4) $-\infty \leqslant p \leqslant q \leqslant r \leqslant s \leqslant +\infty$,

(5) $A(x) = 0$ for $x < p$,

(6) $A(x) \leqslant A(y)$ for $x < y,\ x, y \in [p,q),\ A(q) \geqslant \min(A_-(q),A_+(q))$,

(7) $A(x) = \sup A$ for $x \in (q,r)$,

(8) $A(x) \geqslant A(y)$ for $x < y,\ x, y \in (r,s],\ A(r) \geqslant \min(A_-(r),A_+(r))$,

(9) $A(x) = 0$ for $x > s$.

Theorem 2. A is an (SCV)-interval iff there exists $r \in \overline{\mathbb{R}}$ such that

(10) $A(r) \geqslant \min(A_-(r),A_+(r))$,

(11) $A(x) < A(y)$ for $x < y < r$,

(12) $A(x) > A(y)$ for $y > x > r$.

In the above conditions the values $A(-\infty)$ and $A(+\infty)$ are meant as $\limsup A(x)$ for $x \to -\infty$ or $x \to +\infty$, respectively.

4. Comparison theorems. Now we compare different definitions of fuzzy interval.

Theorem 3. For any A we have

(13) (PM) $\Rightarrow$ (UM) $\Rightarrow$ (CV) .

Proof. If A is a (PM) interval, then for any $a \in [q,r]$ we get property (UM). If A is an (UM)-interval, then for any $x, y, z \in \mathbb{R}$, $x < y < z$ we have

(14) $A(z) \geqslant A(x) \geqslant \min(A(x),A(y))$ for $z < a$, $A(z) \geqslant A(y) \geqslant \min(A(x),A(y))$ for $z \geqslant a$

and we obtain (1), so A is a (CV)-interval.

The converse implications in (13) do not hold. For example, the characteristic function of open crisp interval fulfils (UM) but does not fulfil (PM). Now putting

$$A(x) = \begin{cases} 0 & \text{for } x \leqslant 0 , \\ e^{-x} & \text{for } x > 0 \end{cases}$$

we get the (CV)-interval which is not an (UM)-interval.

Theorem 4. For any A we have

(15) (SUM) $\Rightarrow$ (SCV) ,

(16) (SUM) $\Rightarrow$ (SPM) .

Proof. Let A be a (SUM)-interval. Now the strong version of inequalities (14) leads us to the property (SCV), so we have (15). Putting $p = -\infty$, $q = r = a$ and $s = +\infty$ we obtain also (SPM), i.e. (16) holds.

Considering the (SCV)-interval

$$A(x) = \begin{cases} \frac{1}{2}e^{x} & \text{for } x \leqslant 0 , \\ e^{-x} & \text{for } x > 0 \end{cases}$$

we see that the converse implication in (15) does not hold. It is also obvious that any (SPM)-interval with $p \neq -\infty$ or $q \neq r$ or $s \neq +\infty$ is not a (SUM)-interval. In the same way we see that properties (SPM) and (SCV) are non-comparable.

Considering a continuous A we obtain the equivalences in (13) and (15) but such an assumption is too strong, because it excludes the characteristic functions of crisp intervals. However, assumption of upper semicontinuity admits the characteristic functions of closed crisp intervals and we get

Theorem 5. For any A we have

(17) (UC,PM) $\Leftrightarrow$ (UC,UM) $\Leftrightarrow$ (UC,CV) ,

(18) (UC,SUM) $\Leftrightarrow$ (UC,SCV) .

Proof. If A is an upper semicontinuous then there exists $x_0 \in \overline{\mathbb{R}}$ such that

(19) $$A(x_0) = \sup_{x \in \mathbb{R}} A(x) .$$

Now, by Theorem 1, any (UC,CV)-interval fulfils (PM) and using Theorem 3 we obtain the equivalence (17).

If A is an (UC,SCV)-interval, then putting $a = x_0$ in (19) we get the property (SUM) by Theorem 2, which - together with (15) - gives (18).

5. Fuzzy numbers. If we consider fuzzy number as a special case of fuzzy interval with singleton core (cf. Dubois, Prade [6]), then by Theorem 5 we obtain three equivalent definitions of fuzzy number. Now, the condition (CV) is not convenient, because it does not distinguish the core point. So we can use the property (PM) with $q = r$ but the simplest formula is implied by (UM):

$$\exists!_{a \in \mathbb{R}} \left(\forall_{x < y \leqslant a} A(x) \leqslant A(y) , \quad \forall_{a \leqslant x < y} A(x) \geqslant A(y) \right) .$$

Under this definition any crisp number (precisely: the characteristic function of a singleton in ℝ) is a special case of fuzzy number. In this case the arithmetic of fuzzy numbers contains the real number arithmetic (cf. Goetschel, Voxman [7]).

Another approach to the definition of fuzzy number is proposed by McCain [10] but we do not discuss it here.

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