# Fuzzy-set spaces with Hausdorff's metric: Completeness, separability, embedding

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## Preliminaries

Let $n$ be a positive integer. Denote by $R^n$ the $n$-dimentional vector space with Euclidean inner product $\langle\cdot,\cdot\rangle$ and Euclidean norm $|\cdot|$. Let $x \in R^n$ and $\gamma \in [0,\infty)$. Denote by $Q^n$ the subset of $R^n$ consisting of all vectors with rational coordinates. Let $i, j = 1, 2, \dots$.

Denote by $\mathcal{U}$ the set of all functions $\mu : R^n \to [0,1]$. The elements of $\mathcal{U}$ are known as characteristic functions of fuzzy subsets of $R^n$. An addition $+$, a multiplication by $\gamma$ and an order $\leq$ on the set $\mathcal{U}$ are defined as usual [1], by generalizing the Minkowski operations and the by-inclusion order for ordinary sets. Let also $\sigma$ denote the characteristic function of the closed unit ball $S^*$ (about the origin) in the space $R^n$. Let $\nu \in \mathcal{U}$ and $x^* \in S^*$.

## 1. Space of closed fuzzy sets

Consider a function that belongs to $\mathcal{U}$, takes the value 1 at least once, has the bounded support and is upper semi-continuous. Denote by $\mathcal{L}$ the set of all such functions. A fuzzy set whose characteristic function belongs to $\mathcal{L}$ will be referred to as non-empty bounded closed. Let $\mu, \mu_i \in \mathcal{L}$.

Let

$$\rho(\mu_1, \mu_2) = \min\{\gamma : \mu_1 \leq \mu_2 + \gamma 6, \mu_2 \leq \mu_1 + \gamma 6\}.$$

The number $\rho(\mu_1, \mu_2)$ will be called the Hausdorff distance between fuzzy sets with characteristic functions $\mu_1$ and $\mu_2$. The function $\rho$ will be called the Hausdorff metric on the set $\mathcal{L}$.

Proposition. The Hausdorff distance between any two fuzzy sets with characteristic functions from $\mathcal{L}$ is equal to the supremum of ordinary Hausdorff's distances [2] between their same-level cuts, where the supremum is taken over the set $(0, 1]$ of levels. The proof is immediate.

Theorem. $(\mathcal{L}, \rho)$ is a complete non-separable metric space; the algebraic operations and the order on the set $\mathcal{L}$ being coordinated with the metric $\rho$. In particular, the limit $\mu$ of any fundamental sequence $\{\mu_i\}$ has the form

$$\mu = \inf_i \mathrm{cl} \sup_j \mu_{i+j}, \text{ where } \mathrm{cl}\, \nu = \inf\{\mu : \nu \leq \mu\}.$$

The proof is immediate but not too short.

## 2. Space of closed convex fuzzy sets

Consider a function that belongs to $\mathcal{L}$ and is quasi-concave. Denote by $\mathcal{L}O$ the set of all such functions. A fuzzy set whose characteristic function belongs to $\mathcal{L}O$ will be referred to as non-empty bounded closed convex.

Consider also a function $\varphi : S^* \to R^1$, for which

$$|\varphi(x^*)| \leq const\, |x^*| ,$$

where $const$ depends on $\varphi$ (but doesn't on $x^*$). Denote by $\mathcal{F}$ the set of all such functions. As usual, equip the $\mathcal{F}$ with the pointwise addition, multiplication by scalars, order and the norm

$$\|\varphi\| = \sup \{ |\varphi(x^*)| / |x^*| : x^* \neq 0 \} .$$

Theorem. $(\mathcal{L}O, \rho)$ is a complete non-separable metric space. It can be isometrically embedded in the space $\mathcal{F}$, the algebraic operations and the order being also preserved, by the formula $\mu \to \varphi = \mu^*$, where

$$\mu^*(x^*) = \max \{ \langle x, x^* \rangle : \mu(x) \geq |x^*| \} .$$

The proof is immediate (except for, perhaps, the completeness; in the case of confusion see Item 5). The function $\mu^*$ generalizes the notion of an ordinary-set support function [3].

## 3. Space of closed crisp sets

Consider a function that belongs to $\mathcal{L}$ and takes only two values: $0$ and $1$. Denote by $\mathcal{LR}$ the set of all such functions. A fuzzy set whose characteristic function belongs to $\mathcal{LR}$ will be referred to as non-empty bounded closed crisp.

Theorem. $(\mathcal{LR}, \rho)$ is a complete metric space. For the set of all finite subsets of $Q^n$, the set of their characteristic functions is countable and dense in $(\mathcal{LR}, \rho)$.

In fact, metric properties of the space $\mathcal{LR}$ are well known [2].

## 4. Space of balls

Consider a function of the form $\gamma 6$, for some $\gamma$. Denote by $\mathcal{E}$ the set of all such functions. The support of a function $\gamma 6$ is called the ball of radius $\gamma$ about the origin in the space $R^n$.

Theorem. $(\mathcal{E}, \rho)$ is isometric to $[0, \infty)$; the algebraic operations and the order being also preserved.

The proof is obvious.

5. Regular fuzzy-set spaces

A subspace $\mathcal{V}$ of $\mathcal{U}$ will be called regular if it is closed relative to the $\inf$-operation and contains $\sigma$. For example, the spaces $\mathcal{L}$, $\mathcal{LO}$, $\mathcal{LR}$, $\mathcal{E}$ are regular.

Theorem. Let a subspace $\mathcal{V}$ of $\mathcal{U}$ be regular and be contained in $\mathcal{L}$. Then the space $(\mathcal{V}, \rho)$ is complete.

The proof is based on properties of the projector

$$cv\,\mu = \inf\{\nu \in \mathcal{V} : \mu \leq \nu\}$$

onto the regular subspace $\mathcal{V}$ of the ordered semi-linear space $\mathcal{L}$. Adduce them:

$cv\,\mu$ exists and belongs to $\mathcal{V}$,

$$\mu \in \mathcal{V} \Rightarrow cv\,\mu = \mu,$$

$$\mu \leq cv\,\mu,$$

$$\mu_1 \leq \mu_2 \Rightarrow cv\,\mu_1 \leq cv\,\mu_2,$$

$$cv\,(\mu_1 + \mu_2) \leq cv\,\mu_1 + cv\,\mu_2,$$

$$cv\,(\gamma\mu) = \gamma\, cv\,\mu,$$

and hence

$$\rho\,(cv\,\mu_1, cv\,\mu_2) \leq \rho\,(\mu_1, \mu_2).$$

## Comments

A complete Russian version of the paper exists in the deposit form [5] and consists of 28 typescript pages. Moreover, ideas of Item 2 are used in the paper [6] to develop an elementary theory of differential equations in the space $\mathcal{L}_0$ of fuzzy sets. Remark also the related paper [4].

## References


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