ON INDEXED FUZZY SUBSETS

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Introduction. In 1965 L. A. Zadeh defined a fuzzy set by extending the usual set characteristic function to an infinite valued one, [2]. The fuzzy subsets theory provides a methodology and mathematical apparatus more adequate than the crisp ones (compare e.g. [1]). So, in a quite natural way an idea appears to develop a theory of fuzzy economical systems. But Zadeh's theory does not allow to take a specific properties of economic processes into consideration. Taking into consideration these specific properties of economic systems which we will not discuss here we introduce and develop basic ideas of the indexed fuzzy subsets.

In the first part of this paper we define indexed t-norm, t-conorm and indexed negation, which are used as set operators for indexed fuzzy sets. In the second and third part we give the fundamental definitions and properties of indexed fuzzy subsets. We present only this theory of indexed fuzzy subsets which will be necessary for our further considerations on economical systems.

1. Indexed triangular norms.

Let T be a subset of real numbers R and let $G_T^{\langle 0,1\rangle}$ denotes the family of all functions $g: T \to \langle 0,1\rangle$.

DEFINITION 1.1. A function $TN : G_T^{\langle 0,1\rangle} \times G_T^{\langle 0,1\rangle} \to G_T^{\langle 0,1\rangle}$ is called indexed triangular norm (indexed t-norm for short) iff

T1: - if $g^1, g^2 \in G_T^{\langle 0,1\rangle}$ and $g^1(t) = g^2(t) = 0$ $(t \in T)$ then $TN(g^1,g^2)(t) = 0$,

- if $g^1, g^2 \in G_T^{\langle 0,1\rangle}$ and $g^2(t) = 1$ $(t \in T)$ then $TN(g^1,g^2)(t) = TN(g^2,g^1)(t) = g^1(t)$;

T2: if $g^1, g^2, \bar{g}^1, \bar{g}^2 \in G_T^{\langle 0,1\rangle}$ and $g^1(t) \leq \bar{g}^1(t)$, $g^2(t) \leq \bar{g}^2(t)$ $(t \in T)$ then $TN(g^1,g^2)(t) \leq TN(\bar{g}^1,\bar{g}^2)(t)$;

T3: $TN(g^1,g^2) = TN(g^2,g^1)$, $g^1, g^2 \in G_T^{\langle 0,1\rangle}$;

T4: $TN(TN(g^1,g^2),g^3) = TN(g^1,TN(g^2,g^3))$, $\forall\, g^1,g^2,g^3 \in G_T^{\langle 0,1\rangle}$.

DEFINITION 1.2. A function $CTN : G_T^{\langle 0,1\rangle} \times G_T^{\langle 0,1\rangle} \to G_T^{\langle 0,1\rangle}$, which is defined as t-norm except that the condition T1 is is changed into

CT1 : - if $g^1, g^2 \in G_T^{\langle 0,1\rangle}$ and $g^1(t) = g^2(t) = 1$ $(t \in T)$ then $CTN(g^1,g^2)(t) = 1$,

- if $g^1, g^2 \in G_T^{\langle 0,1\rangle}$ and $g^2(t) = 0$ $(t \in T)$ then $CTN(g^1,g^2)(t) = CTN(g^2,g^1)(t) = g^1(t)$

is called indexed triangular conorm (indexed t-conorm for short).

DEFINITION 1.3. An indexed negation is a mapping

$C : G_T^{\langle 0,1\rangle} \to G_T^{\langle 0,1\rangle}$ such that

C1: $C(0_f) = 1_f$, $0_f, 1_f \in G_T^{\langle 0,1\rangle}$, $\forall t \in T$ $0_f(t) = 0$, $1_f(t) = 1$;

C2: $C(C(f)) = f$;

C3: C is strictly decreasing and continuous.

If $TN(\cdot,\cdot)$ is an indexed t-norm, then $CTN(\cdot,\cdot) = C(TN(C(\cdot),C(\cdot)))$ is an indexed t-conorm and conversely $TN(\cdot,\cdot) = C(CTN(C(\cdot),C(\cdot)))$.

Examples of indexed t-norms and indexed t-conorms are:

$$TN_0(g^1,g^2) = \min(g^1,g^2)$$
$$CTN_0(g^1,g^2) = \max(g^1,g^2) \qquad (1.1)$$

$$TN_1(g^1,g^2) = g^1 \cdot g^2$$
$$CTN_1(g^1,g^2) = g^1 + g^2 - g^1 \cdot g^2 \qquad (1.2)$$

$$TN_\infty(g^1,g^2) = \max(g^1 + g^2 - 1_f\ ,\ 0_f)$$
$$CTN_\infty(g^1,g^2) = \min(g^1 + g^2\ ,\ 1_f) \qquad (1.3)$$

$$TN_w(g^1,g^2)(t) = \begin{cases} \min(g^1(t),g^2(t)) & \text{if } \max(g^1(t),g^2(t)) = 1, \\ 0 & \text{if } \max(g^1(t),g^2(t)) < 1 \end{cases}$$
$$CTN_w(g^1,g^2)(t) = \begin{cases} \max(g^1(t),g^2(t)) & \text{if } \min(g^1(t),g^2(t)) = 0, \\ 1 & \text{if } \min(g^1(t),g^2(t)) > 0 \end{cases} \qquad (1.4)$$

It is immediately seen that for each indexed t-norm TN we have:

$$TN_w(g^1,g^2) \leq TN(g^1,g^2) \leq TN_0(g^1,g^2)\ .$$

2. Fundamental definitions and properties.

Let Y denotes arbitrary, but for further considerations fixed set. Next $\mathcal{P}(Y)$ denotes the family of all non-void subsets of Y.

Let F be a mapping from T to $\mathcal{P}(Y)$. So, $\forall t \in T \quad F(t) \subset Y$. Instead of F(t) we will write F_t.

DEFINITION 2.1. A generalized Cartesian product of the sets F_t $(t \in T)$, F(T) say, is the set of all functions $f : T \to Y$ such that $f(t) \in F_t$, $\forall t \in T$.

DEFINITION 2.2. An index fuzzy subset, v say, is a function $v \in G_T^{\langle 0,1 \rangle}$.

DEFINITION 2.3. An indexed fuzzy subset of $F(T)$, A_v say, is a mapping $A_v : F(T) \to G_T^{\langle 0,1 \rangle}$ such that :

(i) if $v(t)=0$ then $A_v(f)(t)=0$, $\forall f \in F(T)$, $\forall t \in T$;

(ii) if there exists an element $t \in T$ such that $f^1(t)=f^2(t)$ then $A_v(f^1)(t)=A_v(f^2)(t)$, $f^1, f^2 \in F(T)$.

Example 1. Let $T=\{t \in R : t \geqslant 0\}$ and let for all $t \in T$ $F_t = R$. Let us define an index fuzzy subset v in the following way:

$$v(t) = \begin{cases} t & \text{if } t \in \langle 0,1 \rangle , \\ -t+2 & \text{if } t \in (1,2 \rangle , \\ 0 & \text{otherwise.} \end{cases}$$

Now, let us consider the following indexed fuzzy subset A_v :

$$A_v(f)(t) = v(t) \wedge |f(t)| \quad , \quad f \in F(T), \ t \in T .$$

Now, we are going to formulate another definition of indexed fuzzy subset.

DEFINITION 2.4. Let A_t $(t \in T)$ be a fuzzy subset of F_t. An indexedfuzzy subset, A_v say, is a mapping

$A_v : F(T) \to G_T^{\langle 0,1 \rangle}$ such that

$A_v(f)(t) = A_t(f(t)) * v(t)$, $f \in F(T)$, $t \in T$,

where $*$ denotes an operation such that:

- if $A_t(f(t)) \cdot v(t)=0$ then $A_v(f)(t)=0$.

Example 2. Let T, F_t and v are defined as in Example 1. Now, let us consider the following fuzzy subsets A_t $(t \in T)$:

$\forall x_t \in F_t$

$$A_t(x_t) = \begin{cases} t \wedge |x_t| & \text{if } t \in \langle 0,1 \rangle , \\ (-t+2) \wedge |x_t| & \text{if } t \in (1,2 \rangle , \\ 0 & \text{otherwise.} \end{cases}$$

Let us define an indexed fuzzy subset in the following way:

$\forall f \in F(T)$ and $\forall t \in T$

$A_v(f)(t) = A_t(f(t)) \wedge v(t)$.

DEFINITION 2.5. An indexed fuzzy subset is called empty, $\emptyset$ say, if $\forall f \in F(T)$ and $\forall t \in T$ $A_v(f)(t) = 0$.

Let v' and v'' are the index fuzzy subsets.

DEFINITION 2.6. The indexed fuzzy subsets $A_{v'}$ and $A_{v''}$ are equal, $A_{v'} = A_{v''}$ say, if $\forall f \in F(T)$ $A_{v'}(f) = A_{v''}(f)$.

DEFINITION 2.7. An indexed fuzzy subset $A_{v'}$ contain an indexed fuzzy subset $A_{v''}$, $A_{v''} \subset A_{v'}$ say, if $\forall f \in F(T)$ $A_{v''}(f) \leq A_{v'}(f)$.

DEFINITION 2.8. A union of two indexed fuzzy subsets $A_{v'}$ and $A_{v''}$ is the indexed fuzzy subset, $A_{v'} \cup A_{v''}$ say, such that $\forall f \in F(T)$

$A_{v'} \cup A_{v''}(f) = CTN(A_{v'}(f), A_{v''}(f))$.

DEFINITION 2.9. An intersection of two fuzzy subsets $A_{v'}$ and $A_{v''}$ is the indexed fuzzy subset, $A_{v'} \cap A_{v''}$ say, such that $\forall f \in F(T)$

$A_{v'} \cap A_{v''}(f) = TN(A_{v'}(f), A_{v''}(f))$.

DEFINITION 2.10. Complement of an indexed fuzzy subset A_v is the indexed fuzzy subset, CA_v say, such that $\forall f \in F(T)$ and $\forall t \in T$

$$CA_v(f)(t) = \begin{cases} Co(A_v(f)(t) & \text{if } v(t) \neq 0, \\ 0 & \text{if } v(t) = 0, \end{cases}$$

where Co is a negation.

We now write down some immediate and useful consequences of Definitions 2.8, 2.9 and 2.10.

- $\emptyset \cup A_v = A_v$,
- $\emptyset \cap A_v = \emptyset$,
- $A_{v'} \cup A_{v''} = A_{v''} \cup A_{v'}$,
- $A_{v'} \cap A_{v''} = A_{v''} \cap A_{v'}$, (commutative laws)
- $A_{v'''} \cup (A_{v''} \cup A_{v'}) = (A_{v'''} \cup A_{v''}) \cup A_{v'}$
- $A_{v'''} \cap (A_{v''} \cap A_{v'}) = (A_{v'''} \cap A_{v''}) \cap A_{v'}$ (associative laws)
- De Morgan Laws obviously hold if we model intersection,

complementation respectively by an indexed t-norm TN, an indexed negation C, and the indexed t-conorm CTN $CTN = C(TN(C(\cdot), C(\cdot)))$ for union. We then have $TN = C(CTN(C(\cdot), C(\cdot)))$ which indicates that we can start from indexed t-conorm and an indexed negation as-well.

3. Algebraic operations on indexed fuzzy subsets. Convex indexed fuzzy subsets.

Let us consider a generalized set F(T) such that for any $t \in T$ F_t are some linear reference spaces.

DEFINITION 3.1. For any two indexed fuzzy subsets $A_{v'}$ and $A_{v''}$ by $A_{v'} + A_{v''}$ an indexed fuzzy subset is understood whose membership function is related to those of $A_{v'}$ and $A_{v''}$ by

$$A_{v'} + A_{v'}(f) = \sup_{f' + f'' = f} TN(A_{v'}(f'), A_{v''}(f'')),$$

for any $f \in F(T)$.

DEFINITION 3.2. For any two indexed fuzzy subsets $A_{v'}$ and $A_{v''}$ by $A_{v'} \cdot A_{v''}$ an indexed fuzzy subset is understood whose membership function is related to those of $A_{v'}$ and $A_{v''}$ by

$$A_{v'} \cdot A_{v''}(f) = \sup_{f' \cdot f'' = f} (A_{v'}(f') \cdot A_{v''}(f'')),$$

for any $f \in F(T)$.

DEFINITION 3.3. Let A_v be some indexed fuzzy subset of F(T) and λ some function from T to R. By $\lambda \cdot A_v$ such an indexed fuzzy subset of F(T) is understood that

$$A_v(f)(t) = \begin{cases} A_v(\frac{f}{\lambda})(t) & \text{if } \forall t' \in T \ \ \lambda(t') \neq 0, \\ \sup\limits_{f' \in F(T)} A_v(f')(t) & \text{if } f(t) = 0 \text{ and } \lambda(t) = 0, \\ 0 & \text{if } \lambda(t) = 0 \text{ and } f(t) \neq 0, \\ A_v(\frac{f}{\lambda})(t) & \text{if } \lambda(t) \neq 0 \text{ and there exists } t' \in T \\ & \quad \text{such that } \lambda(t') = 0, \text{ where} \end{cases}$$

$$\lambda'(\bar{t}') = \begin{cases} \lambda(\bar{t}') & \text{if } \bar{t}' \neq t', \\ 1 & \text{if } \bar{t}' = t'. \end{cases}$$

DEFINITION 3.4. An indexed fuzzy subset $A_v \subset F(T)$ is called convex if $\forall f^1, f^2 \in F(T)$ and $\forall a,b > 0$ such that $a + b = 1$

$$A_v(af^1 + bf^2) \geq A_v(f^1) \wedge A_v(f^2).$$

DEFINITION 3.5. An indexed fuzzy subset $K_v \subset F(T)$ is called indexed fuzzy cone if $\forall f \in F(T)$ and $\forall a > 0$

$$K_v(af) = K_v(f).$$

THEOREM 3.1. An indexed fuzzy subset K_v is a convex indexed fuzzy cone iff

(i) $\forall f \in F(T)$ and $\forall a > 0 \quad K_v(af) = K_v(f)$,

(ii) $\forall f^1, f^2 \in F(T) \quad K_v(f^1 + f^2) \geq K_v(f^1) \wedge K_v(f^2)$.

Veritably, let K_v be an indexed fuzzy cone. Then taking into account definitions 3.4 and 3.5 we observe that

$$K_v(f^1 + f^2) = K_v(a(1-a)f^1 + (1-a)af^2) \geq K_v((1-a)f^1) \wedge K_v(af^2) = K_v(f^1) \wedge K_v(f^2).$$

So (ii) holds.

Now, let the conditions (i) and (ii) be satisfied. Then K_v is an indexed fuzzy cone and $\forall f^1, f^2 \in F(T)$ and $\forall a \in (0,1)$ we get:

$$K_v(af^1 + (1-a)f^2) \geq K_v(af^1) \wedge K_v((1-a)f^2) = K_v(f^1) \wedge K_v(f^2).$$

So, K_v is an indexed convex fuzzy cone.

THEOREM 3.2. If $A_{v'}, A_{v''} \subset K_v$ with some given convex indexed fuzzy cone, then $A_{v'} + A_{v''} \subset K_v$.

Indeed, taking into account Theorem 3.1 we see that for any $f^1, f^2 \in F(T)$ such that $f^1 + f^2 = f \in F(T)$ we have

$$K_v(f) \geq K_v(f^1) \wedge K_v(f^2).$$

But for any indexed t-norm TN we have

$$TN(K_v(f^1), K_v(f^2)) \leq K_v(f^1) \wedge K_v(f^2).$$

Hence

$$K_v(f) \geq TN(K_v(f^1), K_v(f^2)).$$

So, with respect to Definition 3.1 , if $f^1 + f^2 = f \in F(T)$ then we have

$$K_v(f) \geq \sup_{f^1+f^2=f} TN(K_v(f^1), K_v(f^2)) \geq \sup_{f^1+f^2=f} TN(A_{v'}(f^1), A_{v''}(f^2)) = A_{v'} + A_{v''}(f).$$

For a given $f^o \in F(T)$ and an indexed fuzzy subset A_v we define the indexed fuzzy subset $\{f^o\}_{A_v}$ setting

$$\{f^o\}_{A_v}(f) = \begin{cases} A_v(f) & \text{if } f = f^o, \\ 0_f & \text{if } f \neq f^o, \end{cases}$$

$f \in F(T)$.

THEOREM 3.3. If K_v is a convex indexed fuzzy cone then

$$\{f^1\}_{K_v} + \{f^2\}_{K_v} \subset \{f^1 + f^2\}_{K_v} .$$

Indeed, from Theorem 3.1 for $f = f^1 + f^2$ we have

$$\{f^1 + f^2\}_{K_v}(f) \geq TN(K_v(f^1), K_v(f^2)) = \{f^1\}_{K_v} + \{f^2\}_{K_v}(f).$$

For the remaining f the inclusion is evident.

THEOREM 3.4. If $\{f^1\}_{K_v} \subset A_{v'}$ and $\{f^2\}_{K_v} \subset A_{v''}$, then

$$\{f^1\}_{K_v} + \{f^2\}_{K_v} \subset A_{v'} + A_{v''} .$$

Actually, taking into account Definition 3.1 we see that for $f = f^1 + f^2$ there holds:

$$\{f^1\}_{K_v} + \{f^2\}_{K_v}(f) = TN(K_v(f^1), K_v(f^2)) \leq TN(A_{v'}(f^1), A_{v''}(f^2)) \leq \sup_{f^1+f^2=f} TN(A_{v'}(f^1), A_{v''}(f^2)) = A_{v'} + A_{v''}(f).$$

Let $G_v^{\langle 0,1\rangle}$ be a family of all functions $g : T \to \langle 0,1\rangle$ which satisfies the following conditions:

- if $v(t) = 0$ then $g(t) = 0$,
- if $v(t) > 0$ then $g(t) > 0$, $t \in T$.

DEFINITION 3.6. Let $r \in G_v^{\langle 0,1\rangle}$. The r-cut of the indexed fuzzy subset A_v is the (crisp) set

$A_v^r = \{f \in F(T) : A_v(f) \geq r\}$.

THEOREM 3.5. An indexed fuzzy subset A_v is convex iff for any function $r \in G_v^{\langle 0,1\rangle}$ the set A_v^r is convex.

Proof. Let us assume that $\forall\, r \in G_v^{\langle 0,1\rangle}$ the set A_v^r is convex. Let $A_v(f^2) \geq A_v(f^1) = r$. Then $f^2 \in A_v^r$ and $af^1 + (1-a)f^2 \in A_v^r$ $(a \in (0,1))$. So,

$A_v(af^1 + (1-a)f^2) \geq r = A_v(f^1) = A_v(f^1) \wedge A_v(f^2)$.

Now, let A_v be a convex indexed fuzzy subset and let $r \in G_v^{\langle 0,1\rangle}$, $f^1 \in F(T)$, $A_v(f^1) = r$. Then A_v^r we can define as a set of all functions $f^2 \in F(T)$ such that $A_v(f^2) \geq A_v(f^1)$. Because A_v is convex so, each function $af^1 + (1-a)f^2$, $(a \in (0,1))$ belongs to A_v^r. Hence A_v^r is convex.

4. Remarks.

Let us note that if T is a singleton set (for example $T = \{\bar{t}\}$), then the definitions 1.1, 1.2 and 1.3 are the classical definitions of t-norm, t-conorm and negation respectively.

Now, let us assume that $T = \{\bar{t}\}$, $v(\bar{t}) = 1$ and $*$ denotes the operation such that

$A_t(f(t)) * v(t) = A_t(f(t))$.

Then the definitions 2.3 and 2.4 become the definition of fuzzy subset in Zadeh's sense.

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References

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