A THEOREM ON IMPLICATION FUNCTIONS DEFINED FROM TRIANGULAR NORMS

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ABSTRACT : Several transformations which enable implication functions in multi-valued logics to be generated from conjunctions have been proposed in the literature. It is proved that for a rather general class of conjunctions modeled by triangular norms, the generation process is closed, thus shedding some light on the relationships between seemingly independent classes of implication functions.

0 - INTRODUCTION

It is now well-known (Alsina et al [1], Dubois [2], Prade [7]) that a good model of a fuzzy set-theoretic intersection, or equivalently of a conjunction function in multivalued logics, is a triangular norm (Menger [6], Schweizer and Sklar [8]). Using a De-Morgan-like transformation yields the corresponding model of a fuzzy set-theoretic union or multivalued disjunction function. Let * be a triangular norm, modeling a conjunction, and n a negation function. If P, Q ... are propositions, whose degrees of truth are v(P), v(Q) ... then

$$v(P \cap Q) = v(P) * v(Q)$$

$$v(\neg P) = n(v(P))$$

Several techniques for deriving implication functions from conjunctions have been proposed, in the past, namely (e.g. Valverde [10])

a) $v(P \to Q) = v(\neg(P \wedge \neg Q)) = n(v(P) * n(v(Q)))$

b) $v(P \to Q) = \sup\{s,\ s \in [0,1],\ v(P) * s \leq v(Q)\}$

The motivation of a) is clear. Transformation b) defines a pseudo-complement in a Brouwerian lattice, and is related to the following identity in set theory :

$$\bar{A} \cup B = \overline{A \setminus B} = \bigcup\{S,\ A \cap S \subseteq B\}$$

where $A \setminus B$ is a set-difference and the bar stands for complementation. Although apparently unrelated, these two classes of implication functions, and a third one obtained by contraposition can be generated by the same processes if the class of conjunction operations from which they stem is enlarged. This is the topic of this paper. After a background on triangular norms and their functional representation is recalled, the main theorem is expressed and proved. The introduced transformations are applied to the basic t-norms.

1 - BACKGROUND

A triangular co-norm $\perp$ (t-co-norm for short) is a two-place function from $I \times I$ to I (where I denotes the real interval $[0,1]$) such that

i) $\forall\ (a, b) \in I^2,\ a \perp b = b \perp a$

ii) $\forall\ (a,b,c) \in I^3,\ (a \perp b) \perp c = a \perp (b \perp c)$

iii) if $a \leq b$ and $c \leq d$, then $a \perp c \leq b \perp d$

iv) $\forall\ a \in I,\ 0 \perp a = a$

v) $1 \perp 1 = 1$

A continuous triangular co-norm $\perp$ such that

vi) $\forall\ a \in]0,1[,\ a \perp a > a$ (Archimedean property)

can be expressed in terms of a generator φ which is a continuous strictly increasing function from I to $[0, +\infty]$, with $\varphi(0) = 0$, under the form (Ling [5]) :

$$a \perp b = \varphi^*(\varphi(a) + \varphi(b)) \qquad (1)$$

where φ^* is the pseudo-inverse of φ, defined by

$$\varphi^*(a) = \begin{cases} \varphi^{-1}(a) \text{ if } a \in [0,\ \varphi(1)] \\ 1 \text{ if } a \in [\varphi(1),\ +\infty) \end{cases} \qquad (2)$$

if $\varphi(1) < +\infty$, the t-co-norm $\perp$ is said to be nilpotent ; otherwise the t-norm is said to be strict.

Note that

$$\varphi^* \circ \varphi = \text{id.} \qquad (3)$$

while $\varphi \circ \varphi^* \neq$ id. Moreover φ is defined up to a positive multiplicative constant λ, i.e. φ and $\lambda\varphi$, $\lambda > 0$, generate same the co-norm.

By definition, a strong negation function n is a continuous strictly decreasing function from I to I, such that

vii) $\forall a \in I$, $n(n(a)) = a$

viii) $n(0) = 1$

Trillas [9] proved that any strong negation function n can be generated from a continuous strictly increasing function φ from I to $[0, +\infty)$, such that $\varphi(0) = 0$ and $\varphi(1) < +\infty$, under the form

$$n(a) = \varphi^{-1}(\varphi(1) - \varphi(a)) \qquad (4)$$

To a t-co-norm $\perp$ and a strong negation function n, is associated a two-place function from I^2 to I defined by

$$a * b = n(n(a) \perp n(b)) \qquad (5)$$

(5) expresses a n-duality since then $a \perp b = n(n(a) * n(b))$; * is called a triangular norm ; it satisfies, i, ii, iii, and

ix) $\forall a, \quad 1 * a = a$

x) $0 * 0 = 0$

instead of iv and v respectively ; $\perp$ satisfies vi if and only if * satisfies

xi) $\forall a \in]0,1[, a * a < a$

Then * and $\perp$ are said to be Archimedean.

Any t-norm * is such that

$$\forall (a, b) \in I^2, T_W(a, b) = \begin{cases} a \text{ if } b = 1 \\ b \text{ if } a = 1 \\ 0 \text{ otherwise} \end{cases} \leq a * b \leq \min(a, b) \qquad (6)$$

while, for any t-co-norm $\perp$,

$$\forall (a, b) \in I^2, \max(a, b) \leq a \perp b \leq T_W^*(a, b) = \begin{cases} a \text{ if } b = 0 \\ b \text{ if } a = 0 \\ 1 \text{ otherwise} \end{cases} \qquad (7)$$

Note that min (resp. max) is a continuous t-norm (resp. t-co-norm) which does not satisfy xi (resp. vi), and T_W (resp. T_W^*) is a non-continuous t-norm (resp. t-co-norm).

2 - THEOREM

For any two-place operation $\odot$ on I, let us define the four following transformations

- $\mathcal{R}(\odot)$ is the two-place operation on I defined by

$$a[\mathcal{R}(\odot)]\, b = b \odot a \qquad (8)$$

- $\mathcal{J}_n(\odot)$ is the two-place operation on I defined by

$$a[\mathcal{J}_n(\odot)]\, b = n(a \odot n(b)) \qquad (9)$$

where n is a strong negation function. Clearly $\mathcal{J}_n \circ \mathcal{J}_n = id.$

- $\mathcal{C}(\odot)$ is the two-place operation on I defined by

$$a[\mathcal{C}(\odot)]\, b = \sup\{s,\ s \in [0,1],\ a \odot s \leq b\} \qquad (10)$$
$$= 0 \text{ if } \nexists\, s,\ a \odot s \leq b$$

- $\mathcal{V}_n(\odot)$ is the two-place operation on I defined by

$$a[\mathcal{V}_n(\odot)]\, b = n(b) \odot n(a) \qquad (11)$$

Clearly $\mathcal{V}_n = \mathcal{J}_n \circ \mathcal{R} \circ \mathcal{J}_n$. $\mathcal{J}_n$ and $\mathcal{C}$ express transformations a) and b).

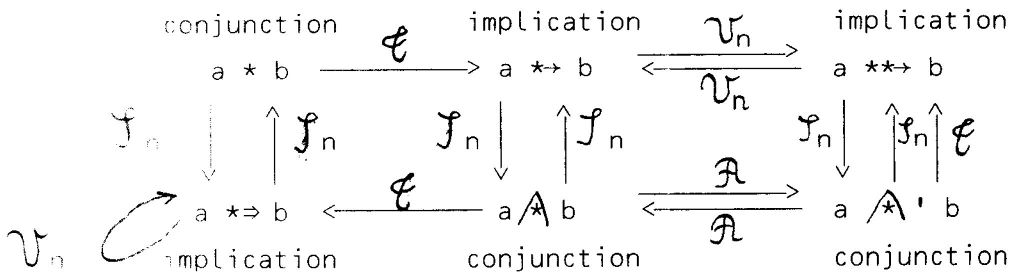
Now, we can prove the following theorem, stated without proof in (Dubois & Prade, [4]) and partially in (Dubois & Prade [3]) :

If $*$ is the t-norm min, or if $*$ is a continuous Archimedean t-norm, then, we have

$$- \mathcal{C} \circ \mathcal{J}_n \circ \mathcal{C}(*) = \mathcal{J}_n(*) \qquad (12)$$
$$- \mathcal{C} \circ \mathcal{R} \circ \mathcal{J}_n \circ \mathcal{C}(*) = \mathcal{V}_n \circ \mathcal{C}(*) \qquad (13)$$

where n is a strong negation.

The contents of this theorem are sketched on the following diagram where notations are introduced for the different operations derived from a triangular norm $*$.



Moreover if $*$ is issued from a nilpotent t-co-norm $\perp$ by (5), then $\mathcal{T} \circ \mathcal{J}_n$ = id., i.e. $* = \overset{*}{\wedge} = \overset{*}{\wedge}'$ and $*\Rightarrow \; = \; *\!\!\to \; = \; **\!\!\to$, provided that n and $\perp$ have the same generator φ.

3 - PROOF

3.1 - Proof of (12) (continuous Archimedean t-norms)

The proof is given for slightly more general 2-place functions than continuous Archimedean t-norms, namely we do not assume $\varphi(0) = 0$, i.e. we do not assume axioms [illegible] and ix to be satisfied. Then, $\forall a \in [0, \varphi(0)]$, $\varphi^*(a) = 0$.

$$
\begin{aligned}
a * b &= n(n(a) \perp n(b)) \\
&= n[\varphi^*(\varphi(n(a)) + \varphi(n(b)))]
\end{aligned}
\qquad (14)
$$

Then

$$a[\mathcal{T}(*)]b = \sup\{s,\ s \in [0,1],\ \varphi^*(\varphi(n(a)) + \varphi(n(s))) \geq n(b)\}$$

Since $\varphi(n(a)) + \varphi(n(s)) \geq \varphi(0)$, the following reasoning is valid :

$$
\begin{aligned}
&\varphi^*(\varphi(n(a) + \varphi(n(s))) \geq n(b) \\
\Leftrightarrow\ &\varphi(n(s)) \geq \varphi(n(b)) - \varphi(n(a)) \\
\Leftrightarrow\ &s \leq n[\varphi^*(\varphi(n(b)) - \varphi(n(a)))] \text{ if } a \geq b \ ;
\end{aligned}
$$

if $a \leq b$, $\forall s$, $\varphi(n(s)) \geq 0 \geq \varphi(n(b)) - \varphi(n(a))$. Thus, we get :

$$
a[\mathcal{T}(*)]b = a *\!\!\to b = \begin{cases} n[\varphi^*(\varphi(n(b)) - \varphi(n(a)))] & \text{if } a \geq b \\ 1 & \text{if } a \leq b \end{cases}
\qquad (15)
$$

Then

$$
a[\mathcal{J}_n \circ \mathcal{T}(*)]b = a \overset{*}{\wedge} b = \begin{cases} \varphi^*(\varphi(b) - \varphi(n(a))) & \text{if } a \geq n(b) \\ 0 & \text{if } a \leq n(b) \end{cases}
\qquad (16)
$$

Then

$$a[\mathcal{T} \circ \mathcal{J}_n \circ \mathcal{T}(*)]b = a *\!\Rightarrow b = \sup\{s,\ s \in [0,1],\ a \overset{*}{\wedge} s \leq b\}$$

$$= \max[\sup\{s,\ s \in [0,1], n(s) \geq a\},\ \sup\{s,\ s \in [0,1], n(s) \leq a,\ \varphi^*(\varphi(s) - \varphi(n(a))) \leq b\}]$$

First $\sup\{s,\ s \in [0,1],\ n(s) \geq a\} = n(a)$;

second, $\varphi^*(\varphi(s) - \varphi(n(a))) \leq b$

$$
\begin{aligned}
&\Rightarrow \varphi(s) - \varphi(n(a)) \leq \varphi(b) \text{ if } \varphi(s) \geq \varphi(n(a)) + \varphi(0) \\
&\Rightarrow s \leq \varphi^*(\varphi(n(a)) + \varphi(b))
\end{aligned}
$$

Let us check that the condition $n(s) \leq a$ is satisfied for this choice of s, i.e. $n(\varphi^*(\varphi(n(a)) + \varphi(b))) \leq a$.

This inequality is equivalent to $\varphi^*(\varphi(n(a)) + \varphi(b)) \geq n(a)$ which obviously holds since $\varphi(n(a)) + \varphi(b) \geq \varphi(n(a))$ and $\varphi^* \circ \varphi = \mathrm{id}$.

Now, if $\varphi(s) \leq \varphi(n(a)) + \varphi(0)$, then $s \leq \varphi^*(\varphi(n(a)) + \varphi(0)) \leq \varphi^*(\varphi(n(a)) + \varphi(b))$, $\forall\, b \in [0,1]$. Thus we get

$$
\begin{aligned}
a \ast\!\Rightarrow b &= \max[n(a), \varphi^*(\varphi(n(a)) + \varphi(b))] \\
&= \varphi^*(\varphi(n(a)) + \varphi(b)) \qquad (17) \\
&= a[\mathcal{S}_n(*)]b
\end{aligned}
$$

Note that $\mathcal{V}_n(\ast\!\Rightarrow) = \ast\!\Rightarrow$. <u>Q.E.D.</u>

3.2 - <u>Proof of (13)</u> (Continuous Archimedean t-norms)

We proceed with the same relaxed assumptions as the proof of (12). It is easy to figure out that

$$
a[\mathcal{R} \circ \mathcal{S}_n \circ \mathcal{C}(*)]\, b = a \mathbin{\mathcal{A}'} b = \begin{cases} \varphi^*(\varphi(a) - \varphi(n(b)) & \text{if } n(b) \leq a \\ 0 & \text{if } n(b) \geq a \end{cases} \qquad (18)
$$

Then

$$
a[\mathcal{C} \circ \mathcal{R} \circ \mathcal{S}_n \circ \mathcal{C}(*)]b = a \ast\!\ast\!\!\to b = \sup\{s,\ s \in [0,1],\ a \mathbin{\mathcal{A}'} s \leq b\}
$$

$$
= \max[n(a), \sup\{s,\ s \in [0,1],\ n(s) \leq a,\ \varphi^*(\varphi(a) - \varphi(n(s))) \leq b\}]
$$

Note that $\varphi^*(\varphi(a) - \varphi(n(s))) \leq b$

$\Leftrightarrow \varphi(a) - \varphi(n(s)) \leq \varphi(b)$ if $\varphi(a) - \varphi(0) \geq \varphi(n(s))$

$\Leftrightarrow s \leq n[\varphi^*(\varphi(a) - \varphi(b))]$ if $a \geq b$

If $a \leq b$, $\forall\, s$, $\varphi(n(s)) \geq 0 \geq \varphi(a) - \varphi(b)$;

We can check that $n[n[\varphi^*(\varphi(a) - \varphi(b))]] \leq a$, if $a \geq b$.

Indeed this is equivalent to $\varphi^*(\varphi(a) - \varphi(b)) \leq a$

which obviously holds since $\varphi(a) - \varphi(b) \leq \varphi(a)$ and $\varphi^* \circ \varphi = \mathrm{id}$;

Now, if $\varphi(a) - \varphi(0) \leq \varphi(n(s))$, then

$s \leq n \circ \varphi^*(\varphi(a) - \varphi(0)) \leq n \circ \varphi^*(\varphi(a) - \varphi(b))$, $\forall\, b \in [0,1]$.

Thus we get

$$a \ast\ast\!\!\to b = \begin{cases} \max[n(a), n[\varphi^\star(\varphi(a) - \varphi(b))]] & \text{if } a \geq b \\ 1 & \text{if } a \leq b \end{cases}$$

but $a \geq \varphi^\star(\varphi(a) - \varphi(b))$ and finally

$$a \ast\ast\!\!\to b = \begin{cases} n[\varphi^\star(\varphi(a) - \varphi(b))] & \text{if } a \geq b \\ 1 & \text{if } a \leq b \end{cases} \tag{19}$$

$$= n(b) \ast\!\!\to n(a) = a[\mathcal{V}_n \circ \mathcal{E}(\ast)]\ b$$

Q.E.D.

3.3 - Particular case : nilpotent t-norms

It is assumed that $(\perp, \ast, n)$ are based on the same generator φ, i.e. $n(a) = \varphi^\star(\varphi(1) - \varphi(a))$, $a \perp b = \varphi^\star(\varphi(a) + \varphi(b))$, $a \ast b = n(n(a) \perp n(b))$. It is easy to figure out that $\varphi \circ n(a) = \varphi(1) - \varphi(a), \forall\ a$. Hence
$a \ast\!\!\Rightarrow b = \varphi^\star(\varphi(n(a)) + \varphi(b)) = \varphi^\star(\varphi(1) - \varphi(a) + \varphi(b))$.

Besides, if $a \geq b$

$n[\varphi^\star(\varphi(n(b)) - \varphi(n(a)))] = n[\varphi^\star(\varphi(a) - \varphi(b))]$

$= \varphi^\star(\varphi(1) + \varphi(b) - \varphi(a))$

since $a \geq b \Rightarrow \varphi(a) - \varphi(b) \leq \varphi(1)$

and $n \circ \varphi^\star(k) = \varphi^\star(\varphi(1) - k)$ if $k \leq \varphi(1)$

if $a \leq b$ then $\varphi(1) - \varphi(a) + \varphi(b) \geq \varphi(1)$ and $a \ast\!\!\Rightarrow b = 1$

We have proved that $a \ast\!\!\to b = a \ast\!\!\Rightarrow b$.

Now $\ast\ast\!\!\to\ = \mathcal{V}_n(\ast\!\!\to) = \mathcal{V}_n(\ast\!\!\Rightarrow) = \ast\!\!\Rightarrow$

$\hat{\ast} = \mathcal{T}_n(\ast\!\!\to) = \mathcal{T}_n(\ast\!\!\Rightarrow) = \ast$

$\hat{\ast}' = \mathcal{T}_n(\ast\ast\!\!\to) = \mathcal{T}_n(\ast\!\!\Rightarrow) = \ast$.

3.4 - min and T_W

The proof in this case can be obtained by straight-forward check for min. The theorem does not hold for T_W.

Examples

We provide the results of the transformations for the four basic t-norms in the appended table 1, where $n(a) = 1 - a$ in any case.

t-norm *	$T_W(a, b)$	max(a + b - 1, 0)	a . b	min(a, b)
nature	discontinuous	nilpotent	strict	non-Archimedean
co-norm	$T^*_W(a, b)$	min(a + b, 1)	a + b - ab	max(a, b)
$a \overset{*}{\rightarrow} b$	1 - a if b = 0 b if a = 1 1 otherwise	min(1, 1 - a + b)	1 - a + ab	max(1 - a, b)
$a \overset{*}{\leftarrow} b$	1 if a < 1 b if a = 1	min(1, 1 - a + b)	min(1, b/a) (1 if a = 0)	1 if a $\leq$ b b if a > b
$a \overset{**}{\leftarrow} b$	1 if b > 0 1 - a if b = 0	min(1, 1 - a + b)	min(1, $\frac{1-a}{1-b}$) (1 if b = 1)	1 if a $\leq$ b 1 - a if a > b
$a \overset{*}{\wedge}{}' b$	0 if b < 1 a if b = 1	max(a + b - 1, 0)	max(0, $\frac{a+b-1}{b}$) (0 if b = 0)	0 if a + b $\leq$ 1 a if a + b > 1
$a \overset{*}{\wedge} b$	0 if a < 1 b if a = 1	max(a + b - 1, 0)	max(0, $\frac{a+b-1}{a}$) (0 if a = 0)	0 if a + b $\leq$ 1 b if a + b > 1

Table 1 - Conjunctions and implications

4 - CONCLUSION

An important consequence of the theorem is the closure of the generation processes of implication functions from a triangular norm based on transformations a) and b). Iterating these processes does not yield an uncontrollable birth of implication families. This closure property is a strong point in favor of methods a) and b). Moreover we have generated a new kind of conjunction operations $\overset{*}{\wedge}$ which coïncide with classical conjunctions of the proposition calculus, but are not commutative when non-extreme truth values are combined. This lack of commutativity may be natural in some contexts when propositions which do not play the same role have to be combined in a conjunctive manner. For instance when P is to be combined with $P \rightarrow Q$, for the purpose of detaching Q ; or if the concept of time is accounted for, and $P \wedge Q$ underlies the assumption that P was known before Q. So we believe that this new class of conjunctions (and dual disjunctions) does not challenge our intuition. Unfortunately, many of these operations are not associative either.

However, we do not claim that the closure theorem is a complete answer to the problem of generating multivalued implication functions. Several families fall outside our framework : for instance those derived from a conjunction and a negation by means of the identity $P \rightarrow Q = \neg P \vee (P \wedge Q)$; or the implication proposed by Yager [11], based on the power operation : $a \rightarrow b = b^a$!

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