ON THE EXTENSION OF POSSIBILITY MEASURES

Wang Zhenyuan*, Zhang Zhipeng**

*Dept. of Mathematics, Hebei University,
Baoding, Hebei, China

**Dept. of Mathematics, Beijing Normal
University, Beijing, China

ABSTRACT

In this paper, we study a general theory of extension of the possibility measures, and prove that, given arbitrarily a nonempty class $\mathcal{C}$ of sets, any nondecreasing mapping from the class of all of unions of $\mathcal{C}$ to the unit interval may be extended uniquely to a generalized possibility measure on the plump field generated by $\mathcal{C}$.

KEYWORDS

Possibility measure; fuzzy measure; possibility distribution.

INTRODUCTION

The possibility measure, which was introduced by L.A.Zadeh [8], is Sugeno's fuzzy measure (cf. [2],[3],[7]) when it is defined on a finite universe. But, in general, the possibility measure doesn't possess the continuity from above, and therefore, it is not always a fuzzy measure, if the universe is infinite. We may introduce a new concept of semi-continuous fuzzy measure on the measurable space, which is a nonnegative monotone set function possessing the continuity from below and its value at the empty set is zero. Evidently, Sugeno's fuzzy measure is a semi-contonuous fuzzy measure, and the possibility measure is also a semi-continuos fuzzy measure.

It is difficult to establish a general extension theory for Sugeno's fuzzy measures (or for the semi-continuous fuzzy measures), because of losing, in general, the additivity which the classical measures possess. Up to now, the extension was only discussed in some special cases (cf. [1],[4],[6],[9]). In this paper, we'll study a general theory of extension of the possibility measures, and prove that, given arbitrarily a nonempty class $\mathscr{C}$ of sets, any nondecreasing mapping from the class of all of atoms of $\mathscr{C}$ to the unit interval may be extended uniquely to a generalized possibility measure on the plump field generated by $\mathscr{C}$.

In the following, we make the conventions: $\bigcup\{\cdot \mid t\in\phi\} = \phi$, and $\sup_{t\in\phi}\{\cdot\} = 0$.

AS-SYSTEMS AND PLUMP FIELDS

Let U be a universe, $\mathscr{C}$ be a nonempty class of subsets of U.

Definition 1. A nonempty class $\mathscr{F}_p$ of subsets of U is called the plump field, if it is closed under arbitrary intersection and arbitrary union operations.

We denote the plump field generated by $\mathscr{C}$, i.e., the minimun plump field including $\mathscr{C}$, by $\mathscr{F}_p(\mathscr{C})$. It is easy to prove that

$$\mathscr{F}_p(\mathscr{C}) = \{ \bigcup_{t\in T} \bigcap_{s\in S_t} A_s \mid A_s \in \mathscr{C},\ S_t \text{ and } T \text{ are arbitrary index sets} \}.$$

Wang [illegible] introduced the concept of ample field, which is a nonempty class of sets closed under complement and arbitrary union operations. Evidently, if $\mathscr{C}$ is closed under complement operation, then $\mathscr{F}_p(\mathscr{C})$ is an ample field.

The following definition extends the concept of atom given in [5].

Definition 2. For any point u in $\bigcup\mathscr{C}$, the set $\bigcap\{A \mid u\in A\in\mathscr{C}\}$ is called the atom of u of $\mathscr{C}$, and denoted by A(u).

The atoms of $\mathscr{C}$ may not belong to $\mathscr{C}$.

Proposition 1. Any set in $\mathscr{C}$ may be expressed by an union of atoms of $\mathcal{C}$; and moreover, any intersection of sets of $\mathcal{C}$ may be expressed by an union of atoms of $\mathcal{C}$.

Proof. Let $\{A_s : s \in S\}$ be a family of sets in $\mathcal{C}$. We have

$$\bigcap_{s \in S} A_s = \bigcup\{A(u) \mid u \in \bigcap_{s \in S} A_s\}. \blacksquare$$

Proposition 2. Any intersection of atoms may be expressed by an union of atoms.

Proof. Since the atom of $\mathcal{C}$ is the intersection of sets in $\mathcal{C}$, by using Proposition 1, we obtain the conclusion of this proposition. ∎

Denote the class of all atoms of $\mathcal{C}$ by $\mathcal{A}[\mathcal{C}]$. We have

Proposition 3. If $\mathcal{C}$ is closed under arbitrary intersection operation, then $\mathcal{C} \supset \mathcal{A}[\mathcal{C}]$.

Proof. $\forall A(u) \in \mathcal{A}[\mathcal{C}]$,

$$A(u) = \bigcap\{A \mid u \in A \in \mathcal{C}\} \in \mathcal{C}. \blacksquare$$

It is easy to show the converse of Proposition 3 is not true.

Proposition 4.

$$\mathcal{F}_p(\mathcal{C}) = \{\bigcup_{t \in T} A_t \mid A_t \in \mathcal{A}[\mathcal{C}], T \text{ is arbitrary index set}\}.$$

Proof. By using Proposition 1, from the structure of $\mathcal{F}_p(\mathcal{C})$, it is easy to obtain this conclusion. ∎

Proposition 5. $\bigcup\mathcal{A}[\mathcal{C}] = \bigcup\mathcal{C}$.

Proof. It is evident. ∎

Proposition 6. If $A' \in \mathcal{A}[\mathcal{C}]$, $u \in A'$, then $A(u) \subset A'$.

Proof. [illegible] $A(u') = \bigcap\{A \mid u' \in A \in \mathcal{C}\} = \bigcap_{t \in T} A_t$, where $A_t \in \mathcal{C}$, T is an index set. Since $u \in A'$, we have $u \in A_t \in \mathcal{C}$ for all $t \in T$. Therefore, by the definition of atom, $A(u) \subset A_t$ for all $t \in T$. Consequently, $A(u) \subset A'$. ∎

Proposition 7. $\mathcal{A}[\mathcal{A}[\mathcal{C}]] = \mathcal{A}[\mathcal{C}]$.

Proof. On one hand, $\forall A(u) \in \mathcal{A}[\mathcal{C}]$, if $u \in B$ for some $B \in \mathcal{A}[\mathcal{C}]$, we have, by using Proposition 6, $A(u) \subset B$, and therefore,

$$A(u) \subset \bigcap\{B \mid u \in B \in \mathcal{A}[\mathcal{C}]\},$$

Reviewing $A(u) \in \mathcal{A}[\mathcal{C}]$, we have

$$A(u) = \bigcap\{B \mid u \in B \in \mathcal{A}[\mathcal{C}]\},$$

that is to say, $\mathcal{A}[\mathcal{A}[\mathcal{C}]] \supset \mathcal{A}[\mathcal{C}]$. On the other hand,

$\forall B(u) \in \mathcal{A}[\mathcal{A}[\mathscr{C}]]$,

$$B(u) = \bigcap\{A(u') \mid u \in A(u') \in \mathcal{A}[\mathscr{C}]\},$$

where $A(u') = \bigcap\{A \mid u' \in A \in \mathscr{C}\}$. It follows, by using Proposition 6,

$$B(u) = \bigcap\{A \mid u \in A \in \mathscr{C}\} \in \mathcal{A}[\mathscr{C}],$$

thus we have $\mathcal{A}[\mathcal{A}[\mathscr{C}]] \subset \mathcal{A}[\mathscr{C}]$. Consequently,

$$\mathcal{A}[\mathcal{A}[\mathscr{C}]] = \mathcal{A}[\mathscr{C}]. \blacksquare$$

Proposition 8. If $\mathscr{C}' = \{\bigcup_{t\in T} C_t \mid C_t \in \mathscr{C},$ T is arbitrary index set$\}$, then $\mathcal{A}[\mathscr{C}'] = \mathcal{A}[\mathscr{C}]$.

Proof. Obviously. $\blacksquare$

Proposition 9. $\mathcal{A}[\mathscr{F}_p(\mathscr{C})] = \mathcal{A}[\mathscr{C}]$.

Proof. By using Proposition 4, Proposition 7, Proposition 8, it is not difficult to prove this conclusion. $\blacksquare$

Proposition 10. $\mathscr{F}_p(\mathscr{C}) = \mathscr{F}_p(\mathcal{A}[\mathscr{C}])$.

Proof. From the definition of the atom and by using Proposition 1, we may obtain this conclusion. $\blacksquare$

Definition [illegible]. AU-system is a nonempty class $\mathscr{C}$ with anti-closeness under union operation, i.e., $\forall \mathscr{C}' \subset \mathscr{C}$,

$$\bigcup\mathscr{C}' \in \mathscr{C} \Longrightarrow \bigcup\mathscr{C}' \in \mathscr{C}'.$$

Proposition 11. $\mathcal{A}[\mathscr{C}]$ is an AU-system.

Proof. Let $\{A(u) \mid u \in C \in \mathscr{C}\}$ be a family of atoms of $\mathscr{C}$. Denote $B = \bigcup_{u\in C} A(u)$. If $B \in \mathcal{A}[\mathscr{C}]$, then $\exists u_0 \in B$, such that $B = A(u_0)$. From $u_0 \in \bigcup_{u\in C} A(u)$, we have $u_0 \in A(u'_0)$ for some $u'_0 \in C$. Thus,

$$\bigcup_{u\in C} A(u) \supset A(u'_0) \supset B.$$

The other inclusion relation is evident. $\blacksquare$

Proposition 12. Let $\mathscr{C}$ be an AU-system. If $\mathscr{C} \supset \mathcal{A}[\mathscr{C}]$, then

$$\mathscr{C} - \{\phi\} = \mathcal{A}[\mathscr{C}].$$

Proof. Let $B \in \mathscr{C} - \{\phi\}$. From Proposition 1, $\exists\{A(u) \mid u \in C \in \mathscr{C}\} \subset \mathcal{A}[\mathscr{C}]$, such that $B = \bigcup_{u\in C} A(u)$. Since $\mathscr{C}$ is an AU-system, we have

$$\bigcup_{u\in C} A(u) \in \{A(u) \mid u \in C \in \mathscr{C}\}.$$

Then $B \in \mathcal{A}[\mathscr{C}]$. Consequently, $\mathscr{C} - \{\phi\} \subset \mathcal{A}[\mathscr{C}]$. The conclusion of this proposition is followed by $\mathscr{C} \supset \mathcal{A}[\mathscr{C}]$ and $\phi \notin \mathcal{A}[\mathscr{C}]$. $\blacksquare$

Definition 4. A mapping $\pi : \mathcal{C} \longrightarrow [0,1]$ is called the generalized possibility measure on $\mathcal{C}$, if it satisfies the following conditions:

(GP1) $\pi(\phi) = 0$, if $\phi \in \mathcal{C}$;

(GP2) $\forall \{A_t, t \in T\} \subset \mathcal{C}$ with $\bigcup_{t \in T} A_t \in \mathcal{C}$,

$$\pi(\bigcup_{t \in T} A_t) = \mathrm{Sup}_{t \in T}\, \pi(A_t),$$

where T is an arbitrary index set.

The condition (GP2) implies the monotonicity condition:

(GP3) $\forall A_1, A_2 \in \mathcal{C},\ A_1 \subset A_2 \Longrightarrow \pi(A_1) \leq \pi(A_2)$.

If $\mathcal{C} = \mathcal{P}(U)$, the class of all subsets of U, and $\pi(U) = 1$, the generalized possibility measure π is a possibility measure.

Lemma. Let $\mathcal{C}$ be an AU-system. If a mapping $\pi : \mathcal{C} \longrightarrow [0,1]$ satisfies the conditions (GP1) and (GP3), then π is a generalized possibility measure on $\mathcal{C}$.

Proof. Because $\mathcal{C}$ is an AU-system, the condition (GP2) is satisfied evidently. ∎

Theorem. A mapping $\pi : \mathcal{A}[\mathcal{C}] \longrightarrow [0,1]$ satisfying the condition (GP3) is a generalized possibility measure on $\mathcal{A}[\mathcal{C}]$. And it may be extended to a generalized possibility measure on $\mathcal{F}_p(\mathcal{C})$ uniquely.

Proof. The mapping π satisfies the condition (GP1), for $\phi \notin \mathcal{A}[\mathcal{C}]$. By using Proposition 11 and Lemma, π is a generalized possibility measure on $\mathcal{A}[\mathcal{C}]$. Moreover, $\forall B \in \mathcal{F}_p(\mathcal{C})$, by using Proposition 4, B may be expressed by $B = \bigcup_{t \in T} A_t$, where $A_t \in \mathcal{A}[\mathcal{C}]$, T is an index set, [illegible]

$$\pi(B) \triangleq \mathrm{Sup}_{t \in T}\, \pi(A_t).$$

The definition is unambiguous. In fact, if $B = \bigcup_{s \in S} A'_s$, where $A'_s \in \mathcal{A}[\mathcal{C}]$, S is an index set, then $\forall A_t$, $\exists u_t \in A_t$, such that A_t [illegible]. From $u_t \in B = \bigcup_{s \in S} A'_s$, $\exists s_t \in S$, such that $u_t \in A'_{s_t} \in \mathcal{A}[\mathcal{C}]$. Thus [illegible] $\subset A'_{s_t}$. It follows

$$\mathrm{Sup}_{t \in T}\, \pi(A_t) \leq \mathrm{Sup}_{s \in S}\, \pi(A'_s).$$

Analogously, we may prove the converse inequality. Consequently,

$$\mathrm{Sup}_{t \in T}\, \pi(A_t) = \mathrm{Sup}_{s \in S}\, \pi(A'_s).$$

Now, we are going to show this π is a generalized possibility [illegible] $\mathcal{F}_p(\mathcal{C})$):

(1) [illegible] $\phi = \bigcup\{\cdot \mid t \in \phi\}$, we have $\pi(\phi) = \sup_{t\in\phi}\{\cdot\} = 0$, that is, π [illegible] the condition (GP1);

(2) [illegible] $\{B_t \mid t \in T\} \subset \mathcal{F}_p(\mathcal{C})$, $B_t = \bigcup_{s\in S_t} A_s^{(t)}$, where $A_s^{(t)} \in \mathcal{A}[\mathcal{C}]$, $t \in T$, T and S_t [illegible] arbitrary index sets, we have

$$\bigcup_{t\in T} B_t = \bigcup_{t\in T} \bigcup_{s\in S_t} A_s^{(t)} .$$

and therefore,

$$\pi(\bigcup_{t\in T} B_t) = \sup_{s\in S_t,\, t\in T} \pi(A_s^{(t)}) = \sup_{t\in T}[\sup_{s\in S_t} \pi(A_s^{(t)})] = \sup_{t\in T} \pi(B_t).$$

[illegible] π satisfies the condition (GP2).

The [illegible] of extension is evident. ∎

[illegible] normal possibility distribution f on U can determine [illegible] measure on $\mathcal{P}(U)$ uniquely.

[illegible] is the class of all singletons in $\mathcal{P}(U)$. Evidently, [illegible] and $\mathcal{F}_p(\mathcal{C}) = \mathcal{P}(U)$. Since a possibility distribution [illegible] determine uniquely a mapping

$$[illegible]\ \mathcal{A}[\mathcal{C}] \longrightarrow [0,1]$$
$$\{u\} \longmapsto f(u) ,$$

[illegible] the condition (GP3), by above theorem, it determines [illegible] possibility measure on $\mathcal{P}(U)$ uniquely. $\pi(U) = 1$ [illegible] the normalization condition of f. ∎

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