Solution of a Kind of Pansystems Relation Function Equation and Fuzzy Relation Equation

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Fuzzy relation equation was first posed by E.Sanchez , then a method to solve it was put forward by Y.Y.Tsukamoto, and some others by people in /2,3/. The generalized relation equation is an improtant problem in pansystems methodology which has been studied widely in /1,4,5,6/. But it is not known what is the relation between them. This paper reveals some pansystems logic conservation between them, then poses a method to solve them. Using the method to solve fuzzy relation equation, one can avoid doing the work wasted to search the solutions which have been known or do not exist , and the amount of calculation work needed is less than that in all posed before.

Definition 1. We call f a M relation function on $G_1 \times G_2$ iff:

1. f is a mapping from [0,1] to $P(G_1 \times G_2)$.
2. $f(t) \leq f(t')$ if $t' \leq t$.
3. There exists a number $t'(x,y)$ for each $(x,y) \in G_1 \times G_2$ such that $(x,y) \in f(t') - f(t)$ holds for $t > t'(x,y)$.

We denote $F(G_1,G_2)$ by the set of all M relation function on $G_1 \times G_2$, $F'(n,m)$ by that of all fuzzy Matrixes that the order is $n \times m$.

Theorem 1. If $n = |G_1|$, $m = |G_2|$, then $|F(G_1,G_2)| = |F'(n,m)|$.

Proof Let

$$\theta : f'(\in F'(n,m)) \longrightarrow f, \quad f(t) = \{ (x_i,y_j) / u_{f'}(i,j) \geq t \},$$

$$\theta' : f \ (\in F(G_1,G_2)) \longrightarrow f', \quad u_{f'}(i,j) = t'(x_i,y_j).$$

Obviously $\theta(f') \in F(G_1,G_2)$, $\theta'(f) \in F'(n,m)$, $\theta(\theta'(f)) = f$, $\theta'(\theta(f')) = f'$. This completes the proof.

Theorem 2. $(\theta, \vee)$, $(\theta, \wedge)$ and (θ, ϕ) are three isomorphisms between $F'(n,m)$ and $F(G_1,G_2)$.

Definition 2. If f_1', $X' \in F'(1,n)$, $f_2' \in F'(n,m)$, $f_3' \in F'(1,m)$, then we call (I): $X' \circ f_2' = f_3'$ a fuzzy relation equation. If $f_1 \in F(\{z\}, G_1)$, $f_2 \in F(G_1,G_2)$, $f_3 \in F(\{z\}, G_2)$, then we call (A): $X \circ f_2 = f_3$ a relation function equation. We call (a) and (b) are equivalent iff the solution space of (a) equals to that of (b). We call (A) the corresponding equation of (I) iff (A) = Θ(I).

Theorem 3. If (A) is the corresponding equation of (I), then f' is a solution of (I) iff $\Theta(f')$ of (A).

Theorem 4. If Q' is the solution space of (I), Q of the corresponding equation of (I), then $\Theta(Q') = Q$, $\Theta'(Q) = Q'$.

The discussion above shows that the problem of a fuzzy relation equation (I) can be solved through solving the problem of a corresponding relation function equation (A).

Let $f_2' = (a_{ij}) \in F'(n,m)$, $f_3' = (b_{1i}) \in F'(1,m)$, $f_2'(i) = (c^i_{ij})$, $f_3'(i) = (d^i_{1i})$, $i = 1, \ldots, m$, $f_j(i) = \Theta(f_j'(i))$, $j = 1, 2$,

$$c^i_{uv} = \begin{cases} a_{uv}, & v = i, \\ 0, & v \neq i, \end{cases} \qquad b^i_{1v} = \begin{cases} b_{1v}, & v = i, \\ 0, & v \neq i. \end{cases}$$

It is clear that

$$f_j' = \bigcup_{1 \le i \le m} f_j'(i), \quad f_j = \bigcup_{1 \le i \le m} f_j(i), \quad j = 1, 2;$$

(A) and (B): $X \circ f_2(i) = f_3(i)$, $i = 1, \ldots, m$. are equivalent.

Lemma 1. If $X \subset G_1 \times G_2$, f_v, $f_u \subset G_2 \times G_3$, $f_v \le f_u$, then that g is a solution of $X \circ f_u = \emptyset$ implies that g of $X o f_v = \emptyset$, and that g is a solution of $X \circ f_v = G_1 \times G_3$ implies that g of $X \circ f_u = G_1 \times G_3$.

Let c_i and c_i' be two numbers satisfying:

$$a_{uv} < b_{1i} \text{ iff } a_{uv} \le c_i, \quad a_{uv} > b_{1i} \text{ iff } a_{uv} \ge c_i'.$$

Theorem 5. $X \circ f_2(i) = f_3(i)$ and (c): $X \circ (f_2(i))(c_i) = (f_3(i))(c_i)$, $X \circ (f_2(i))(c_i') = (f_3(i))(c_i')$. are equivalent for i from 1 to m.

Theorem 6. If $f_j(i,1) = f_j(i)(c_i)$, $f_j(i,2) = f_j(i)(c_i')$, $i = 1, \ldots, m$, $j = 2, 3$, then $(x_u, y_v) \in f_j(i,2)$ iff $a_{uv} \ge c_i'$, and $(x_u, y_v) \notin f_j(i,2)$ iff $a_{uv} \le c_i$.

Theorem 7. If $b_{1i} > b_{1j}$, $a_{ui} \ge b_{1i}$, $a_{uj} > b_{1j}$,

(a): $\quad X \circ f_2(u,1) = f_3(u,1)$, $u = i, j$.

(b): $X \circ f_2(j,1) = f_3(j,1)$, $X \circ [f_2(i,1) - (x_u, y_i)] = f_3(i,1)$.

then (a) and (b) are equivalent.

Let J be the set of all (u,i) satisfying the condition in Theorem 7, i.e, there exists j such that $b_{1i} > b_{1j}$, $a_{ui} \ge b_{1i}$, $a_{uj} > b_{1j}$.

Theorem 8. If f_2' and f_3' are given in (I), $f_2'' = (a'_{ui})$,

$$a'_{ui} = \begin{cases} 0, & \text{if } a_{ui} < b_{li}, \\ 0, & \text{if } a_{ui} \geq b_{li} \text{ and } (u,i) \in J, \\ a_{ui}, & \text{if } a_{ui} \geq b_{li} \text{ and } (u,i) \notin J, \end{cases}$$

then (I) and (II): $X \circ f_2'' = f_3'$ are equivalent.

Theorem 9. If $b_{li} \geq b_{lj}$ and $a'_{ui} \neq 0$ implies $a'_{uj} \neq 0$, $u \leq m$, (a): $X \circ f_2(v, l) = f_3(v, l)$, $v = 1, 2$; (b): $X \circ f_2(i, l) = f_3(i, l)$; then (a) and (b) are equivalent.

Let J' be the set of all j satisfying the condition in theorem 9, i.e, there exists i such that $b_{li} \geq b_{lj}$ and $a'_{ui} \neq 0$ implies $a'_{uj} \neq 0$, $u \leq m$; J" be the set of all j being the elements of J' and satisfying $a'_{uj} \neq 0$, then $a'_{uj} = b_{lj}$ or there exists a'_{ui} such that $a'_{ui} > b_{li} = b_{lj}$.

Theorem 10. (A) and (B): $X \circ f_2(i,1) = f_3(i,1)$, $i \notin J'$, $X \circ f_2(i,2) = f_3(i,2)$, $i \notin J''$, are equivalent.

Example 1. If (A) is the corresponding equation of (I),

$$f_2' = \begin{pmatrix} 0,9 & 0,2 & 0,6 & 0,3 \\ 0,4 & 0,5 & 0,3 & 0,2 \end{pmatrix}^T, \qquad f'_3 = \begin{pmatrix} 0,6 \\ 0,5 \end{pmatrix}^T,$$

then the solution space of (A) is

$$X(0,6 < t) = (0 \quad \# \quad \# \quad \#),$$
$$X(0,5 < t \leq 0,6) = (t' \quad \# \quad t' \quad \#),$$
$$X(0 \leq t \leq 0,5) = (t' \quad 1 \quad t' \quad \#).$$

where we use $x_2 = 1$ expresses $(z,x_2) \in X(0 \leq t \leq 5)$, $x_2 = \#$ expresses $x_2 = 0$ or 0, $x_1 = 0$ expresses $(z,x_1) \notin X(0,6 < t)$, $x_1 = x_3 = t'$ expresses that $x_1 = 1$ and $x_3 = 1$ at least one holds.

The discussion above gives the method to solve relation equation equation, in fact, the method can used to solve the equation: $X \circ f_2 = f_3$ (f_i satisfying the restrains 1, 2, in the definition 1 and $f_i(0)$ equals to the all space, i.e, $f_2(0) = G_2 \times G_3$, $f_3(0) = G_1 \times G_3$).

The method in /2/ is the simplest one to solve fuzzy relation equation, it can simplifies (I) to the corresponding equation of (II) in theorem 8. But the one in this paper an simpifies (I) to the corresponding equation of (B) in theorem 10, so if we use $\Theta'(f_2)$ ((B): $X \circ f_2 = f_3$) to search the minimum solution by the method in /2/, then we can avoid a part of work wasted to search the same minimum solutions or to get the solution being not a minimum solution.

If $f_2' = \Theta'(f_2) = (a_{ij}) \in F'(n,m')$, $m' \leq m$, $q = |\{ b_{1v} | v = 1, \ldots, m \}|$, I_i ($i = 1, \ldots, q$) is the set such that $b_{1u} = b_{1v}$ holds for every $(u,v) \in I_i^2$ and $b_{1u} > b_{1v}$ holds for every $u \in I_i$, $v \in I_j$ and $i > j$, $g = (b'_{ij}(\cdot)_{ij}) \in F'(n,q)$,

$$b'_{ij} = \begin{cases} 0, \text{ if } a_{ji} = 0 \text{ holds for each } i \in I_i, \\ b_{1i}, \text{ if } a_{ji} \neq 0, \ i \in I_i, \end{cases}$$

$$(\cdot)_{ij} = I'_{ij} = \{ i \mid a_{ji} \neq 0, \ i \in I_i \},$$

we call I'_{ij} a symbol of b'_{ij}.

Theorem 11. If $x_i \in \{ b'_{vi} \mid 1 \leq v \leq q \}$ ($i = 1, \ldots, n$), $\{ I'_{ij} \mid (i,j) \in d \}$ is the set of their symbols, then ($x_1, \ldots, x_n$) is a minimum solution iff for every $i = 1, \ldots, q$ we have

1. $I'_{ij} \not\subset \bigcup_{j \neq j'} I'_{ij'}$, $(i,j) \in d$, $(i,j') \in i \times (\bigcup_{p \leq i} p \circ d)$,
2. $\bigcup I'_{ij} = I'_i$ ($(i,j) \in i \times (\bigcup_{p \leq i} p \circ d)$).

References

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