Pansystems Analysis of The Reliability of Fuzzy Graphs

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Let V be a nonempty finite set, $F: V^2-I \rightarrow [0,1]$ be a mapping from V^2-I to $[0,1]$, then the pair $G=(V,F)$ is called a fuzzy graph, where $I=\{(x,x):x\in V\}$. In this paper, fuzzy graphs are used to represent conmunication networks. Every pair of vertices $(x,y)\in V^2-I$ is called an edge of G. For a given edge (x,y), $F(x,y)$ is considered as the connectivity of this edge. A path of fuzzy graph G is a finite string of different vertices of G. Let x,y be two vertices of G, a path joining x and y is a path which begins with x and ends at y. Denote by $\mathcal{P}$ the collection of all paths of G and by $\mathcal{P}(x,y)$ the collection of all paths joining x and y.

The connectivity function $F': \mathcal{P}\rightarrow[0,1]$ is defined as follows $F'(P)=\prod_{i=1}^{k-1} F(x_i,x_{i+1})$, where $P=\{x_1,x_2,\cdots,x_k\}\in\mathcal{P}$.

The connectivity of two vertices x and y, denoted by $C_G(x,y)$, is defined as $C_G(x,y)=\max\limits_{p\in\mathcal{P}(x,y)} F'(P)$.

The connectivity of fuzzy graph G, denoted by C(G), is defined as $C(G)=\min\limits_{(x,y)\in V^2-I} C\ (x,y)$.

The connectivity of a fuzzy graph serves as a measurement of the ability of G. Usually, we would like the connectivity of G, $C(G)\geqslant\alpha$ for a certain $\alpha\in(0,1)$ to meet our demand. This is, however, not enough. In designing and analyzinga communication network, we always concern with another important problem: Reliability. That is if the network is somewhat damaged, how about the connectivity? What is the characterization of a fuzzy graph such that its connectivity is at least α even if it is damaged in a certain degree?

To answer this question, we introduce the concept of damage degree of fuzzy graphs. By using pansystems analysis of equivalence relations, we present a necessary and sufficient condition under which the connectivity of a fuzzy graph G is at least α even if l edges of G is damaged.

Denote by $G(\alpha)$ the collection of all subgraph of fuzzy graph G with connectivity greater or equal to α, i,e.

$$G(\alpha)=\{G':G'=(V,F'),\ F'\leqslant F \text{ and } C(G')\geqslant\alpha\},$$

where $G=(V,F)$.

We take the number of damaged edges as the measure of the damage degree, define $G[l]$ as follows:

$G[l]=\{G': G'=(V,F)$ and $|\{(x,y): F(x,y)-F'(x,y)>0\}|<l\}$.

Denote by $\mathcal{P}_G^{\alpha}(x,y)$ the collection of all paths of G which joins x and y and with connectivity greater or equal to α.

Definition. A fuzzy graph G is said to be (l,α)-stable if $G[l]\subset G(\alpha)$.

The intuitive idea behind this definition is quite simple. An (l,α)-stable fuzzy graph has the property that so long as the number of damaged edges is less than l, its connectivity remains greater or equal to α. It ensures the reliability of a fuzzy graph.

Theorem 1. If for every pair of vertices x,y, there exist at least l edge disjoint paths which belongs to $\mathcal{P}_G^{\alpha}(x,y)$, then G is an (l,α)-stable fuzzy graph.

Here two paths $P_1=\{x_1, x_2,\cdots,x_k\}$, $P_2=\{y_1, y_2,\cdots, y_m\}$ is said to be edge disjoint if $x_i=y_j$ implies $x_{i+1}\neq y_{j+1}$. A set of paths is said to be edge disjointed if every two paths in this set are edge disjoint.

However, it can be shown by an example that the condition in theorem 1 is not a necessary condition.

To give a necessary and sufficient condition for a fuzzy graph to be an (l,α)-stable fuzzy graph, we introduce some notations of pansystems methodology.

Let X be a nonempty set, $E[X)$ denotes the set of all equivalence relations on X. If $\delta\in E[X)$, $x\in X$, then $x\circ\delta$ is a subset of X defined as:

$$x\circ\delta=\{y: y\in X, (x,y)\in\delta\}.$$

If $\delta\in E[X)$, then X/δ denotes a quotient set of X, which is defined as a collection of disjoint subsets of X, i.e.

$$X/\delta=\{x\circ\delta: x\in X\}.$$

The norm of δ, denoted by $\|\delta\|$, is defined as $\|\delta\|=X/\delta$

Theorem 2. Let $G=(V,F)$ be a fuzzy graph, G is an (l,α)-stable fuzzy graph if and only if for every pair of vertices $x,y\in V$, $\mathcal{P}_G^{\alpha}(x,y)\neq\emptyset$, and $\delta\in E[\mathcal{P}_G^{\alpha}(x,y))$ with $\|\delta\|\leq l-1$ implies that $\exists P_0\in\mathcal{P}_G^{\alpha}(x,y)$, such that

$$\bigcap_{P\in P_0\circ\delta} P_e=\emptyset.$$

where P_e is the edge expression of P, i.e. if $P=\{x_1,x_2,\cdots, x_k\}$, then $P_e=\{(x_1,x_2),(x_2,x_3),\cdots,(x_{k-1},x_k)\}$.