COMPARISON OF FUZZY NUMBERS AND FUZZY OPIMIZATION

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Inequality relation between two fuzzy numbers is investigated. Certain type of such relation is proposed, its correspondence with the usual inclusion relation and its possible interpretation is discussed. For a special, sufficiently large class of fuzzy numbers /called R-fuzzy numbers/ the introduced relation can be substituted by a couple of ordinary inequalities. This fact may be taken advantage of in fuzzy optimization of problems with linear constraints.

1. Preliminaries

A fuzzy subset of the real line E_1 is called <u>fuzzy number.</u> A fuzzy number $\tilde{a}$ is characterized by its membership function

$$\mu_{\tilde{a}} : E_1 \longrightarrow [0,1] .$$

To distinquish a fuzzy number from a non-fuzzy number we´ll always denote it with a wave " ~ " .

For $\beta \in [0,1]$ we define a β-level set of $\tilde{a}$ as follows:

$$(\tilde{a})^{\beta} = \{x \in E_1 \ ; \ \mu_{\tilde{a}}(x) \geqq \beta\} \ .$$

A fuzzy number $\tilde{a}$ is said to be convex if for any $\beta \in [0,1]$ $(\tilde{a})^{\beta}$ is a convex subset of E_1.

Let $\tilde{a}$, $\tilde{b}$ be fuzzy numbers, $x \in E_1$. Fuzzy numbers $\tilde{c} = \tilde{a}\,x$, $\tilde{d} = \tilde{a} \oplus \tilde{b}$ are defined by the membership functions

/3/ $$\mu_{\tilde{c}}(t) = \max\{0 \ , \sup_{t=ux} \mu_{\tilde{a}}(u)\},$$

/4/ $$\mu_{\tilde{d}}(t) = \sup_{t=u+v} \min\{\mu_{\tilde{a}}(u), \mu_{\tilde{b}}(v)\}$$

respectively. This definition is based on the well-known Extension principle, see e.g. [2]

2. Inequality relation

When comparing two fuzzy numbers, different approaches may be used. In [2] or [3] the relation of inequality between two fuzzy numbers was considered as inclusion relation " $\subseteq$ " in the sense of Zadeh, i.e. $\tilde{a} \subseteq \tilde{b}$ if

/5/ $$\mu_{\tilde{a}}(t) \leqq \mu_{\tilde{b}}(t) \qquad \text{for all } t \in E_1 .$$

Notice that /5/ is equivalent to the condition

/6/ $$(\tilde{a})^{\beta} \subseteq (\tilde{b})^{\beta} \qquad \text{for all } \beta \in [0,1] .$$

Definition 1. Let $\tilde{a}, \tilde{b}$ be fuzzy numbers. The relation " $\precsim$ " between $\tilde{a}$ and $\tilde{b}$, i.e.

$$/7/ \qquad \tilde{a} \precsim \tilde{b}$$

is defined by the formula

$$/8/ \qquad \sup (\tilde{a})^{\beta} \leq \sup (\tilde{b})^{\beta} \quad \text{for all} \quad \beta \in [0,1] \, .$$

Remark 1. Evidently, formula /8/ is equivalent to the following one

$$/9/ \qquad \forall u \in E_1 \; \exists v \in E_1 \; [u \leq v \; \& \; \mu_{\tilde{a}}(u) \leq \mu_{\tilde{b}}(v)] \, .$$

Remark 2. With regard to /6/ it is easy to see that the inclusion relation $\tilde{a} \subseteq \tilde{b}$ implies the couple of relations

$$\tilde{a} \precsim \tilde{b} \quad \text{and} \quad (-1)\tilde{a} \precsim (-1)\tilde{b} \, ,$$

where $(-1)\tilde{a}$, $(-1)\tilde{b}$ are fuzzy numbers defined by /3/ . Evidently, the opposite implication holds for convex fuzzy numbers $\tilde{a}, \tilde{b}$. Thus, from this point of view Definition 1 suggests a less restrictive relation than the formula /5/ does.

Remark 3. To throw more light on the problem of interpretation of the inequality relation introduced by /8/ suppose that /1/ presents e.g. the system of constraints in a linear programming problem of finding an optimal production plan with fuzzy coefficients $\tilde{a}_{ij}$ and right hand sides $\tilde{b}_i$. Then $\tilde{a}_{ij}$ represents the specific consumption of the i-th source for the j-th production activity, $\tilde{b}_i$ the supply level of the i-th source /membership functions of $\tilde{a}_{ij}$, $\tilde{b}_i$ being based e.g. on estimates of several experts/. The inequality

$$/10/ \qquad \tilde{a}_{i1} x_1 \oplus \tilde{a}_{i2} x_2 \oplus \ldots \oplus \tilde{a}_{in} x_n \precsim \tilde{b}_i$$

expresses in this case that the actual consumption of the i-th source will be covered by the existing supply. To each of possible values $u \in E_1$ of the consumption there namely exists a possible value $v \in E_1$ /see Remark 1/ of the supply with $u \leqq v$ and the grade of membership of v in $\tilde{b}_i$ not less than the grade of membership of u in the left hand side of /10/.

3. R - fuzzy numbers

To obtain an explicit and useful formula for the set of all solutions $x = (x_1, \ldots, x_n)$ of the system

$$\tilde{a}_{i1} x_1 \oplus \tilde{a}_{i2} x_2 \oplus \ldots \oplus \tilde{a}_{in} x_n \precsim \tilde{b}_i, \qquad i=1,2,\ldots,m$$
$$x_j \geq 0, \qquad j=1,2,\ldots,n,$$

we restrict ourselves to a special class of fuzzy numbers specified in the following definition.

Definition 2. Let $R : [0, +\infty) \to [0,1]$, $R(0) = 1$, be a non-increasing function which is not constant on $[0, +\infty)$. By $\mathcal{M}_R$ we denote the set of all fuzzy numbers $\tilde{a}$ membership functions of which have the following property:

There are real numbers $m \in E_1$, $\alpha > 0$ such that

/11/
$$\mu_{\tilde{a}}(t) = R\left(\frac{t-m}{\alpha}\right) \quad \text{for} \quad t \geqq m,$$
$$\mu_{\tilde{a}}(t) = \varphi(t) \quad \text{for} \quad t < m,$$

where $\varphi : E_1 \to [0,1]$ is some function.

A number $\tilde{a} \in \mathfrak{M}_R$ will be called R-fuzzy number.

<u>Notation.</u> For a given function R with properties from Definition 2 we define numbers

/12/
$$\varepsilon_R = \sup \{ u \, ; \, R(u) = R(0) = 1 \} ,$$
$$\delta_R = \inf \{ u \, ; \, u \geqq 0 \, , \, R(u) = \lim_{s \to +\infty} R(s) \} ,$$

/see Fig. 1/. Obviously $\varepsilon_R < +\infty$. According to Definition 2 every fuzzy number $\tilde{a} \in \mathfrak{M}_R$ is characterized by a triad $(\varphi, m, \alpha)_R$. We denote

/14/
$$\tilde{a} \equiv (\varphi, m, \alpha)_R .$$

<u>Assertion 1.</u> Let $\tilde{a}, \tilde{b} \in \mathfrak{M}_R$, $x > 0$, $\tilde{a} \equiv (\varphi, m, \alpha)_R$, $\tilde{b} \equiv (\psi, n, \beta)_R$ and let $\tilde{c} = \tilde{a}x$, $\tilde{d} = \tilde{a} \oplus \tilde{b}$. Then $\tilde{c}, \tilde{d} \in \mathfrak{M}_R$, moreover,

$$\tilde{c} \equiv (\chi, mx, \alpha x)_R , \quad \tilde{d} \equiv (\vartheta, m+n, \alpha+\beta)_R ,$$

where χ, ϑ are suitable functions.

The following assertion presents necessary and sufficient conditions for inequality relation between two R-fuzzy numbers as defined in Definition 1.

Assertion 2. Let $\tilde{a}, \tilde{b} \in \mathfrak{M}_R$, $\tilde{a} \equiv (\varphi, m, \alpha)_R$, $\tilde{b} \equiv (\psi, n, \beta)_R$. Then

/16/ $$\tilde{a} \precsim \tilde{b}$$

if and only if

/17/ $$\varepsilon_R(\alpha - \beta) \leqq n - m,$$

and

/18/ $$\delta_R(\alpha - \beta) \leqq n - m$$

/see Fig. 2/.

Proofs of Assertions 1 and 2 are omitted here and will be publicated later in Fuzzy Sets and Systems.

Remark 4. Notice that Assertion 2 includes the case $\delta_R = +\infty$, while the trivial case $\varepsilon_R = +\infty$ has already been excluded in Definition 2. As usualy, we take

$$a.(\pm\infty) = \pm\infty \text{ for } a > 0, \quad a.(\pm\infty) = \mp\infty \text{ for } a < 0,$$
$$0.(\pm\infty) = 0.$$

Remark 5. For a strictly decreasing function R from Definition 2 it is

$$\varepsilon_R = 0, \quad \delta_R = +\infty.$$

Then the inequalities /17/, /18/ have the form

/22/ $$m \leqq n, \qquad \alpha \leqq \beta.$$

4. Use in fuzzy optimization

In this section the previous results will be utilized for solving the following fuzzy optimization problem: Maximize /minimize/ the real function of n variables

$$/23/ \qquad f\,(x_1, x_2, \ldots, x_n)$$

subject to

$$/24/ \qquad \tilde{a}_{i1} x_1 \oplus \tilde{a}_{i2} x_2 \oplus \ldots \oplus \tilde{a}_{in} x_n \precsim \tilde{b}_i, \quad i=1,2,\ldots,m,$$

$$/25/ \qquad x_j \geqq 0, \quad j=1,2,\ldots,n,$$

with $\tilde{a}_{ij} \in \mathcal{M}_{R_i}$, $\tilde{b}_i \in \mathcal{M}_{R_i}$, $\mathcal{M}_{R_i}$ being defined in Definition 2, $\tilde{a}_{ij} \equiv (\varphi_{ij}, m_{ij}, \alpha_{ij})_{R_i}$, $\tilde{b}_i \equiv (\varphi_i, n_i, \beta_i)_{R_i}$, $i=1,\ldots,m$, $j=1,\ldots,n$.

Applying Assertion 1 and excluding the trivial case $x_1 = x_2 = \ldots = x_n = 0$, we obtain

$$/26/ \qquad \tilde{a}_{i1} x_1 \oplus \ldots \oplus \tilde{a}_{in} x_n \equiv (\vartheta, \sum_{j=1}^{n} m_{ij} x_j, \sum_{j=1}^{n} \alpha_{ij} x_j) \in \mathcal{M}_{R_i} .$$

Due to Assertion 2 the system /24/ with fuzzy inequalities is equivalent to the system of ordinary inequalities

$$/27/ \qquad \varepsilon_{R_i}\Big(\sum_{j=1}^{n} \alpha_{ij} x_j - \beta_i\Big) \leqq n_i - \sum_{j=1}^{n} m_{ij} x_j ,$$

$$/28/ \qquad \delta_{R_i}\Big(\sum_{j=1}^{n} \alpha_{ij} x_j - \beta_i\Big) \leqq n_i - \sum_{j=1}^{n} m_{ij} x_j ,$$

$i=1,\ldots,m$. Consequently, the set of all solutions $x = (x_1, \ldots, x_n)$ of the system /24/, /25/ of $m+n$ inequalities may be expressed by means of $2m+n$ ordinary inequalities as the set of all solutions of the system /27/, /28/, /25/ with non-fuzzy coefficients.

Remark 6. Let R_i be strictly decreasing functions on the interval $[0, +\infty)$, $i=1,\ldots,m$. Then

$$\varepsilon_{R_i} = 0 , \quad \delta_{R_i} = +\infty$$

and inequalities /27/, /28/ can be transformed to the form

$$\sum_{j=1}^{n} m_{ij} x_j \leqq n_i , \quad i=1,\ldots,m ,$$

$$\sum_{j=1}^{n} \alpha_{ij} x_j \leqq \beta_i , \quad i=1,\ldots,m .$$

This result is in correspondence with the similar one obtained by D. Dubois and H. Prade in [2].

Remark 7. Setting

$$R_1 = R_2 = \ldots = R_m = R ,$$

where

$$R(u) = 1 \quad \text{for} \quad u \in [-1,1] ,$$

$$R(u) = 0 \quad \text{for} \quad |u| > 1 ,$$

we have

$$\varepsilon_{R_i} = \delta_{R_i} = 1 , \quad i=1,\ldots,m .$$

In this " interval coefficient " case inequalities /27/, /28/ result in the well-kown relations

$$\sum_{j=1}^{n} \bar{a}_{ij} x_j \leq \bar{b}_i \ , \ i=1,\dots,m .$$

with $\bar{a}_{ij} = m_{ij} + \alpha_{ij}$, $\bar{b}_i = n_i + \beta_i$ being right-hand bounds of interval coefficients.

6. Conclusion

In this paper inequality between two fuzzy numbers has been investigated. Certain type of such relation has been proposed, its correspondence with the usual inclusion relation and its possible interpretation has been discussed. For a special class of fuzzy numbers called R-fuzzy numbers the introduced relation can be substituted by a couple of ordinary inequalities. This fact may be taken advantage of in fuzzy optimization with linear fuzzy constraints.

References

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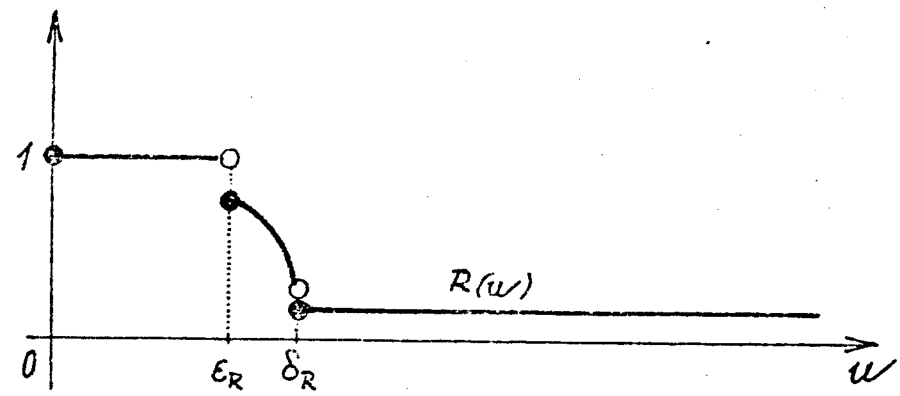

Fig. 1 : R-fuzzy number

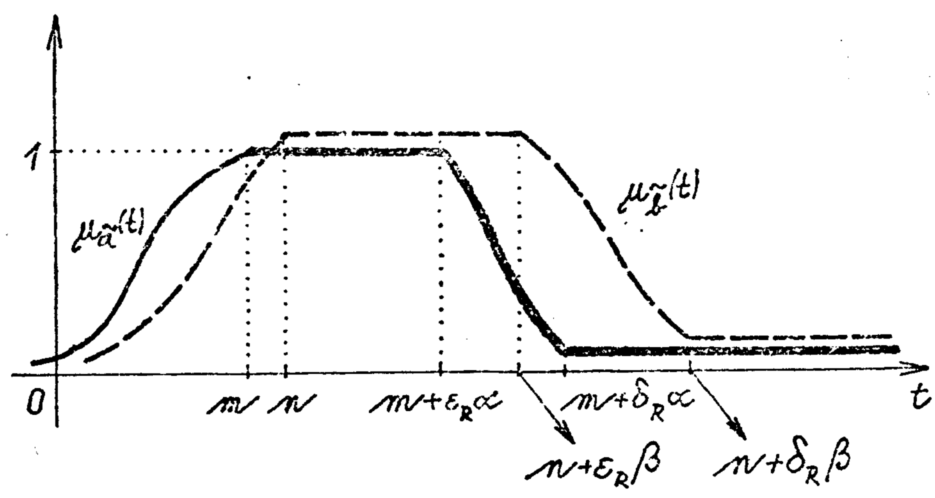

Fig. 2 : Illustration to Assertion 2