On convexity of $\underset{\sim}{S}(\eta_1, \eta_2)$ Projected from random intervals

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Abstract

In this paper the convexity of $\underset{\sim}{S}(\eta_1, \eta_2)$ projected from random intervals is studied, and the sufficient conditions of $\underset{\sim}{S}(\eta_1, \eta_2)$ being convex fuzzy set are given.

1. Preparatory Knowledge

For convenience' sake, in this paper the repeated notations are given as a preparatory knowledge as follows. Their senses remain constant unless they are specified.

η_1, η_2 are independent continuous random variables.

F_1, F_2 are distribution functions of η_1, η_2 respectively.

$p_1(x)$ $p_2(X)$ are distribution densities of η_1, η_2 respectively.

For given two random variables, it is nothing serious that which one is denoted by η_1, therefore η_1 denotes always the random variable the distribution function of which reaches $\frac{1}{2}$ earlier than other one, and the other one is denoted by η_2. If they reach $\frac{1}{2}$ at the same time, any one of the two can be denoted by η_1.

$x_{0\,1}$ is a point that satisfies $F_1(x)=\frac{1}{2}$ if $\{x \mid F_1(x)=\frac{1}{2}\}$ is a singleton set.

$x_{0\,2}$ is a point that satisfies $F_2(X)=\frac{1}{2}$ if $\{x \mid F_2(x)=\frac{1}{2}\}$

is a singleton set.

x_0 is point that satisfies $F_1(x)+F_2(x)=1$ if $\{x \mid F_1(x)+F_2(x)=1\}$ is a singleton set.

x_{0i} is a point that satisfies $p_i(x)=p_2(x)$ if $\{x \mid p_1(x)=p_2(x)\}$ is a singleton set.

Note:

$\delta \triangleq |\frac{1}{2}-F_1(x_0)| = |\frac{1}{2}-F_2(x_0)|$

$\varepsilon_1 \triangleq |F_1(x_1)-\frac{1}{2}|$

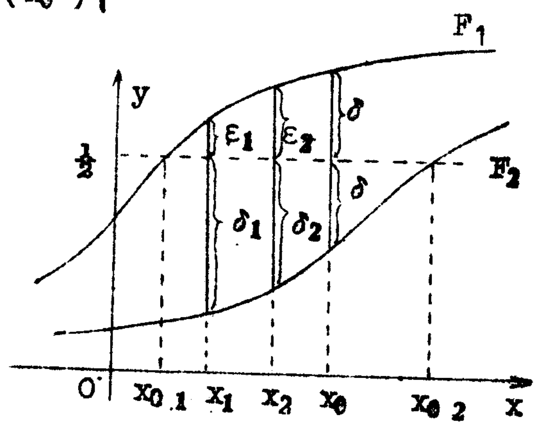
$\varepsilon_2 \triangleq |F_1(x_2)-\frac{1}{2}|$

$\delta_1 \triangleq |F_2(x_1)-\frac{1}{2}|$

$\delta_2 \triangleq |F_2(x_2)-\frac{1}{2}|$

$\Delta\varepsilon \triangleq |\varepsilon_2-\varepsilon_1|$

$\Delta\delta \triangleq |\delta_2-\delta_1|$



Preparatory Theorem 1. Suppose that η_1, η_2 are independent continuous random variables having distribution functions $F_1(x)$ and $F_2(x)$, then

$$\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x)=F_1(x)+F_2(x)-2F_1(x)F_2(x) \qquad (\text{see } [1])$$

According to the definition of a convex fuzzy set [2], it is clear that we have

Preparatory Theorem 2. Suppose that $\underset{\sim}{A} \in \mathcal{F}(R)$ and $\exists X_0 \in R$ such that

$\mu_{\underset{\sim}{A}}(x)$ is nondecreasing if $x < x_0$,

$\mu_{\underset{\sim}{A}}(x)$ is nonincreasing if $x > x_0$;

then $\underset{\sim}{A}$ is a convex fuzzy set.

2. Sufficient Conditions of $\underset{\sim}{S}(\eta_1, \eta_2)$ Being a Convex Fuzzy Set

By the properties of distribution function of continuous random variable, we can prove easily the following two lemmas:

Lemma 1. Suppose that η is a continuous random variable having distribution function $F_\eta(x)$ and set $A=\{x \mid F_\eta(x)=\frac{1}{2}\}$, then A is a nonempty set and A is a closed interval if A is not singleton set.

Lemma 2. Suppose that η is a continuous random variable having distribution function $F_\eta(x)$, then $\{x \mid 0<F_\eta(X)<1\}$ is a open interval. (we regard $(-\infty, +\infty)$ as a particular case of the open interval)

Definition 1. Suppose that η is a continuous random variable. η is said to be strictly increasing if its distribution function $F_\eta(x)$ is strictly increasing on $\{x \mid 0<F(x)<1\}$.

Proposition 1. If η is strictly increasing, there exists one and only one point x_{01} satisfying $F_\eta(x_{01})=\frac{1}{2}$.

The proof is clear.

Lemma 3. If η_1, η_2 are strictly increasing, there exist x_{01} and x_{02} such that

$\mu_{\underset{\smile}{s}(\eta_1, \eta_2)}(x)$ is nondecreasing on $(-\infty, x_{01}]$.

$\mu_{\underset{\smile}{s}(\eta_1, \eta_2)}(x)$ is nonincreasing on $[x_{02}, +\infty)$.

Proof: Since η_1, η_2 are strictly increasing, there exist x_{01} and x_{02} such that $F_1(x_{01})=F_2(x_{02})=\frac{1}{2}$. Due to the restriction of notations of η_1 and η_2 in the preparatory knowledge, we have $x_{01}\le x_{02}$.

$\forall\, x_1<x_2\le x_{01}\le x_{02}$:

$$\mu_{\underset{\smile}{s}(\eta_1, \eta_2)}(x_1)-\mu_{\underset{\smile}{s}(\eta_1, \eta_2)}(x_2)$$

$$=F_1(x_1)+F_2(x_1)-2F_1(x_1)F_2(x_1)-F_1(x_2)-F_2(x_2)+$$
$$+2F_1(x_2)F_2(x_2)$$
$$=(\tfrac{1}{2}-\varepsilon_1)+(\tfrac{1}{2}-\delta_1)-2(\tfrac{1}{2}-\varepsilon_1)(\tfrac{1}{2}-\delta_1)-(\tfrac{1}{2}-\varepsilon_2)-(\tfrac{1}{2}-\delta_2)$$
$$+2(\tfrac{1}{2}-\varepsilon_2)(\tfrac{1}{2}-\delta_2)$$
$$=-\varepsilon_1+\varepsilon_2-\delta_1+\delta_2+\varepsilon_1+\delta_1-\varepsilon_2-\delta_2-2\varepsilon_1\delta_1+2\varepsilon_2\delta_2$$
$$=2(\varepsilon_2\delta_2-\varepsilon_1\delta_1).$$

Since $F_1(x)$ and $F_2(x)$ are nondecreasing, we have $0\le\varepsilon_2\le\varepsilon_1\le\frac{1}{2}$, $0\le\delta_2\le\delta_1\le\frac{1}{2}$.

Thus $\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x_1)-\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x_2)\le 0$,

hence $\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x)$ is nondecreasing on $(-\infty, x_{01}]$.

In the same manner, we have that

$\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x)$ is nonincreasing on $(x_{02}, +\infty)$. ▌

By Lemma3, following Theorem is clear.

Theorem 1. Suppose that η_1, η_2 are strictly increasing and $x_{01}=x_{02}$, then we have

$\underset{\sim}{S}(\eta_1,\eta_2)$ is a convex fuzzy set[2] and

$\max\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x)=\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x_{01})=\frac{1}{2}$.

Lemma 4. Suppose that η_1,η_2 are strictly increasing and $x_{01}<x_{02}$, then

$\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x)>\frac{1}{2}$ on (x_{01},x_{02}).

Proof: $\forall x\in(x_{01}, x_{02})$, we have

$$F_2(x_{01})\le F_2(x)<\tfrac{1}{2}<F_1(x)\le F_1(x_{02}).$$

hence $\varepsilon_1=|F_1(x)-\frac{1}{2}|>0$ and $\delta_1=|F_2(x)-\frac{1}{2}|>0$,

thus $\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x)=F_1(x)+F_2(x)-2F_1(x)F_2(x)$
$$=\tfrac{1}{2}+\varepsilon_1+\tfrac{1}{2}-\delta_1-2(\tfrac{1}{2}+\varepsilon_1)(\tfrac{1}{2}-\delta_1)$$
$$=\tfrac{1}{2}+2\varepsilon_1\delta_1>\tfrac{1}{2}$$ ▌

Lemma 5. Suppose that η_1,η_2 are strictly increasing and $x_{01}<x_{02}$. Set $A=\{x\mid F_1(x)+F_2(x)=1\}$, then we have that

(1) A is a nonempty set.

(2) If A is a singleton set $\{x_0\}$, then $x_{01} < x_0 < x_{02}$.

(3) If A is not a singleton set, then A is a closed interval and $A \subset (x_{01}, x_{02})$.

The proof is trivial and unnecessary.

Lemma 6. Suppose that η_1, η_2 are strictly increasing and $A=\{x \mid F_1(x)+F_2(x)=1\}$ is not a singleton set, let $A=[x', x'']$ (see lemma 5), then we have that

$$F_1(x) \equiv 1 \quad \text{if} \quad x \geq x',$$
$$F_2(x) \equiv 0 \quad \text{if} \quad x \leq x''.$$

Proof: Assume $F_1(x) \not\equiv 1$ if $x \geq x'$, then there exist at least one point $\bar{x} > x'$ such that $\frac{1}{2} < F_1(\bar{x}) < 1$. Hence $F_1(x)$ is strictly increasing and $F_2(x)$ is nondecreasing on $[x', \bar{x}]$, thus there exist at most one point x such that $F_1(x)+F_2(x)=1$. This means that $A=\{x \mid F_1(x)+F_2(x)=1\}$ is at most a singleton set. This contradicts the hypothesis, hence $F(x) \equiv 1$ if $x \geq x'$.

In the same manner, we have

$$F_2(x) \equiv 0 \quad \text{if} \quad x \leq x''. \qquad \blacksquare$$

Theorem 2. Suppose that η_1, η_2 are strictly increasing and $A=\{x \mid F_1(x)+F_2(x)=1\}=[x', x'']$ $(x' < x'')$ then

$\underset{\vee}{S}(\eta_1, \eta_2)$ is a convex fuzzy set, and

$\mu_{\underset{\vee}{S}(\eta_1, \eta_2)}(x) \equiv 1$ on $A=[x', x'']$.

Proof: From lemma 5, it is known that $A=[x', x''] \subset (x_{01}, x_{02})$, namely, $x_{01} < x' < x'' < x_{02}$. And from lemma 6, it is known that $F_1(x) \equiv 1$ if $x \geq x'$ and $F_2(x) \equiv 0$ if $x \leq x''$.

On $[x_{01}, x']$:

Since $\mu_{\underset{\vee}{S}(\eta_1, \eta_2)}(x)=F_1(x)+F_2(x)-2F_1(x)F_2(x)=F_1(x)$, $\mu_{\underset{\vee}{S}(\eta_1, \eta_2)}(x)$ is nondecreasing.

On $[x'', x_{02}]$:

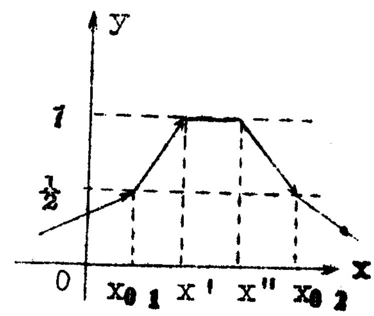
Since $\mu_{\underset{\sim}{S}(\eta_1,\ \eta_2)}(x)=F_1(x)+F_2(x)-2F_1(x)F_2(x)$
$$=1-F_2(x),$$
$\mu_{\underset{\sim}{S}(\eta_1,\ \eta_2)}(x)$ is nonincreasing.

On $[x',\ x'']$: $\mu_{\underset{\sim}{S}(\eta_1,\ \eta_2)}(x)\equiv 1$.

From lemma 3, it is known that:

On $(-\infty,\ x_{01}]$: $\mu_{\underset{\sim}{S}(\eta_1,\ \eta_2)}(x)$ is nondecreasing.

On $(x_{02},\ +\infty)$: $\mu_{\underset{\sim}{S}(\eta_1,\ \eta_2)}(x)$ is nonincreasing.



In summing up, the monotonic cases of $\underset{\sim}{S}(\eta_1,\ \eta_2)$ are illustrated as figure, hence $\underset{\sim}{S}(\eta_1,\ \eta_2)$ is a convex fuzzy set and $\mu_{\underset{\sim}{S}(\eta_1,\ \eta_2)}(x)\equiv 1$ on $[x',\ x'']$. ∎

Lemma 7. Suppose that η_1, η_2 are strictly increasing, and there exists one and only one point x_0 such that $F_1(x_0)+F_2(x_0)=1$.

(1) If $P_1(x)>P_2(x)$ on $(x_{01},\ x_0)$, then $\mu_{\underset{\sim}{S}(\eta_1,\ \eta_2)}(x)$ is strictly increasing on $(x_0,\ x_{02})$.

(2) If $P_1(x)<P_2(x)$ on $(x_0,\ x_{02})$, then $\mu_{\underset{\sim}{S}(\eta_1,\ \eta_2)}(x)$ is strictly decreasing on $(x_0,\ x_{02})$.

Proof: (1) $\forall x_1,\ x_2\in(x_{01},\ x_0)$ and $x_1<x_2$, from preparatory Theorem 1, we have

$$\mu_{\underset{\sim}{S}(\eta_1,\ \eta_2)}(x_1)-\mu_{\underset{\sim}{S}(\eta_1,\ \eta_2)}(x_2)$$
$$=F_1(x_1)+F_2(x_1)-2F_1(x_1)F_2(x_1)-F_1(x_2)-F_2(x_2)+2F_1(x_2)F_2(x_2)$$
$$=\tfrac{1}{2}+\varepsilon_1+\tfrac{1}{2}-\delta_1-2(\tfrac{1}{2}+\varepsilon_1)(\tfrac{1}{2}-\delta_1)-(\tfrac{1}{2}+\varepsilon_2)-(\tfrac{1}{2}-\delta_2)+2(\tfrac{1}{2}+\varepsilon_2)(\tfrac{1}{2}-\delta_2)$$
$$=2(\varepsilon_1\delta_1-\varepsilon_2\delta_2)$$
$$=2[\varepsilon_1(\delta_2+\Delta\delta)-\delta_2(\varepsilon_1+\Delta\varepsilon)]$$
$$=2(\varepsilon_1\Delta\delta-\delta_2\Delta\varepsilon).$$

Since $F_1(x)$ is strictly increasing on (x_{01}, x_0) and $F_2(x)$ is nondecreasing, $0<\varepsilon_1<\delta\leq\delta_2$.

Since $p_1(x) > p_2(x)$ on (x_{01}, x_0), $\Delta\varepsilon>\Delta\delta$.

Hence $\mu_{\underset{\vee}{S}(\eta_1, \eta_2)}(x_1)-\mu_{\underset{\vee}{S}(\eta_1, \eta_2)}(x_2)<0$.

This implies $\mu_{\underset{\vee}{S}(\eta_1, \eta_2)}(x)$ is strictly increasing.

The proof of (2) is similar to (1) so we omit it. ∎

By lemma 3 and lemma 7, we have

Theorem 3. Suppose that η_1, η_2 are strictly increasing and there exists one and only one point x_0. If $p_1(x)>p_2(x)$ on (x_{01}, x_0) and $p_1(x)<p_2(x)$ on (x_0, x_{02}), then $\underset{\vee}{S}(\eta_1, \eta_2)$ is a convex fuzzy set and

$$\max \mu_{\underset{\vee}{S}(\eta_1, \eta_2)}(x)=\mu_{\underset{\vee}{S}(\eta_1, \eta_2)}(x_0)=1-2F_1(x_0)F_2(x_0)$$

Theorem 4. Suppose that η_1, η_2 are strictly increasing and there exists one and only one point x_0, and $p_1(x), p_2(x)$ are continuous on $[x_{01}, x_{02}]$.

(1) If $p_1(x)>p_2(x)$ on $[x_{01}, x_{02}]$, $p_1(x)$ is nonincreasing and $p_2(x)$ is nondecreasing on $[x_0, x_{02}]$, then $\underset{\vee}{S}(\eta_1, \eta_2)$ is a convex fuzzy set.

(2) If $p_1(x)<p_2(x)$ on $[x_{01}, x_{02}]$, $p_1(x)$ is nondecreasing and $p_2(x)$ is nonincreasing on (x_{01}, x_0), then $\underset{\vee}{S}(\eta_1, \eta_2)$ is a convex fuzzy set.

Proof: (1) From lemma 5, it is known that $x_{01}<x_0<x_{02}$.

On $[x_{01}, x_0]$:

Since $p_1(x), p_2(x)$ are continuous and $F_1(x)$, $F_2(x)$ are derivable, by the preparatory theorem, we have

$$\mu_{\underset{\vee}{S}(\eta_1, \eta_2)}(x)=F_1(x)+F_2(x)-2F_1(x)F_2(x),$$

$$[\mu_{\underset{\vee}{S}(\eta_1, \eta_2)}(x)]'=p_1(x)+p_2(x)-2p_1(x)F_2(x)-2p_2(x)F_1(x)$$

$$=p_1(x)[1-2F_2(x)]-p_2(x)[2F_1(x)-1].$$

Since $\frac{2F_1(x)-1}{1-2F_2(x)}$ is nondecreasing and $\frac{2F_1(x_0)-1}{1-2F_2(x_0)}=$

$=\frac{2F_1(x_0)-1}{1-2[1-F_1(x_0)]}=1$, hence $\frac{2F_1(x)-1}{1-2F_2(x)}\leq 1$, namely,

$2F_1(x)-1 \leq 1-2F_2(x)$, and since $p_1(x)>p_2(x)$, we have

$$[\mu_{\underset{\vee}{S}(\eta_1,\eta_2)}(x)]'=p_1(x)[1-2F_2(x)]-p_2(x)[2F_1(x)-1]\geq 0.$$

This implies $\mu_{\underset{\vee}{S}(\eta_1,\eta_2)}(x)$ is nondecreasing on $[x_{01}, x_0]$.

On $[x_0, x_{02}]$:

Since $p_1(x)$ is nonincreasing and $p_2(x)$ is nondecreasing, $\frac{p_1(x)}{p_2(x)}$ is monotone decreasing, and $\frac{p_1(x)}{p_2(x)}>1$.

$\frac{2F_1(x)-1}{1-2F_2(x)}$ is monotone increasing and $\frac{2F_1(x_0)-1}{1-2F_2(x_0)}=1$.

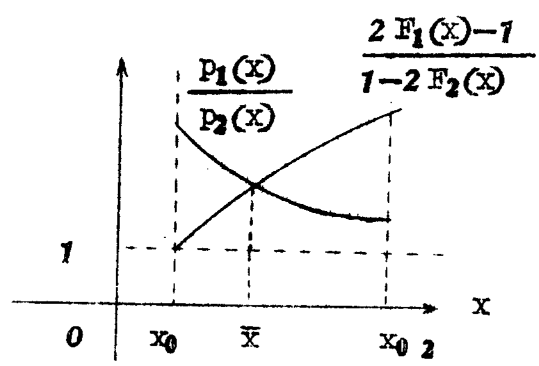
If $\frac{p_1(x)}{p_2(x)}$ and $\frac{2F_1(x)-1}{1-2F_2(x)}$ are non-intersecting, namely,

$\frac{p_1(x)}{p_2(x)}>\frac{2F_1(x)-1}{1-2F_2(x)}$, then $[\mu_{\underset{\vee}{S}(\eta_1,\eta_2)}(x)]'=$

$=p_1(x)[1-2F_2(x)]+p_2(x)[1-2F_1(x)]>0$, hence

$\mu_{\underset{\vee}{S}(\eta_1,\eta_2)}(x)$ is strictly increasing.

If $\frac{p_1(x)}{p_2(x)}$ and $\frac{2F_1(x)-1}{1-2F_2(x)}$ have an intersection point $\bar{x}\in(x_0, x_{02}]$ as shown in Figure on right,



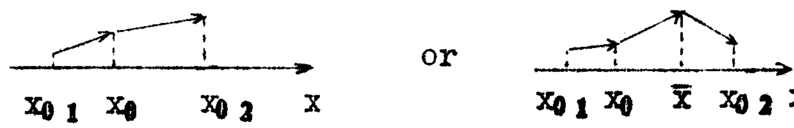
then $\dfrac{p_1(x)}{p_2(x)} \geq \dfrac{2F_1(x)-1}{1-2F_2(x)}$ on $[x_0, \bar{x}]$,

hence $\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x)$ is nondecreasing on $[x_0, \bar{x}]$.

In the same manner, we have

$\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x)$ is nonincreasing on $[\bar{x}, x_{02}]$.

Thus the monotonicities of $\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x)$ on $[x_{01}, x_{02}]$ can only be in the two cases as shown in Figure below.



Referring to lemma 3, we have that $\underset{\sim}{S}(\eta_1,\eta_2)$ is a convex fuzzy set.

The proof of (2) is omitted for it is similar to (1). ∎

Lemma 8. Suppose that $p_1(x)$, $p_2(x)$ are strictly increasing and there exsists one and only one point x_0, $p_1(x)$ and $p_2(x)$ are continuous on $(x_{01}, x_{02}]$, and there exists one and only one point $x_{00} \in (x_{01}, x_{02})$ such that $p_1(x_{00})=p_2(x_{00})$. If $p_1(x) > p_2(x)$ on (x_{01}, x_{00}) and $p_1(x) < p_2(x)$ on (x_{00}, x_{02}), then we have that

(1) $\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x)$ is nondecreasing on $(-\infty, \min\{x_0, x_{00}\})$

(2) $\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x)$ is nonincreasing on $[\max\{x_0, x_{00}\}, +\infty)$

Proof: If $x_{01} < x_0 \leq x_{00} < x_{02}$:

Since $p_1(x) > p_2(x)$ on (x_{01}, x_0), $\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x)$ is nondecreasing on (x_{01}, x_0) (see lemma 7). And by lemma 3, $\mu_{\underset{\sim}{S}(\eta_1,\eta_2)}(x)$ is nondecreasing on $(-\infty, x_0]$, since

$\dfrac{p_1(x)}{p_2(x)} < 1$ and $\dfrac{2F_1(x)-1}{1-2F_2(x)} > 1$ on (x_{00}, x_{02}),

$\frac{p_1(x)}{p_2(x)} < \frac{2F_1(x)-1}{1-2F_2(x)}$, that is,

$p_1(x)[1-2F_2(x)] < p_2(x)[2F_1(x)-1]$. Hence

$[\mu_{\underset{\sim}{S}(\eta_1, \eta_2)}(x)]' < 0$, thus $\mu_{\underset{\sim}{S}(\eta_1, \eta_2)}(x)$ is

nonincreasing on (x_{00}, x_{02}).

And by lemma 3, $\mu_{\underset{\sim}{S}(\eta_1, \eta_2)}(x)$ is nonincreasing on $[x_{00}, +\infty)$.

If $x_{01} < x_{00} < x_0 < x_{02}$:

In the same manner, we have that

$\mu_{\underset{\sim}{S}(\eta_1, \eta_2)}(x)$ is nondecreasing on $(-\infty, x_{00}]$,

$\mu_{\underset{\sim}{S}(\eta_1, \eta_2)}(x)$ is nonincreasing on $[x_0, +\infty)$.

Thus $\mu_{\underset{\sim}{S}(\eta_1, \eta_2)}(x)$ is nondecreasing on $(-\infty, \min\{x_0, x_{00}\}]$,

$\mu_{\underset{\sim}{S}(\eta_1, \eta_2)}(x)$ is nonincreasing on $[\max\{x_0, x_{00}\}, +\infty)$. ▮

From lemma 8, we have

Theorem 5. Suppose that η_1, η_2 are strictly increasing and there exists one and only one point x_0, $p_1(x)$ and $p_2(x)$ are continuous on (x_{01}, x_{02}) and there exists one and only one point x_{00}. If $x_0 = x_{00}$, $p_1(x) > p_2(x)$ on (x_{01}, x_0) and $p_1(x) < p_2(x)$ on (x_0, x_{02}), then $\underset{\sim}{S}(\eta_1, \eta_2)$ is a convex fuzzy set.

Theorem 6. Suppose that η_1, η_2 are strictly increasing and there exists one and only one point x_0 ; $p_1(x), p_2(x)$ are continuous and there exists one and only one point x_{00} $(x_0 \neq x_{00})$ on (x_{01}, x_{02}); $p_1(x) > p_2(x)$ on (x_{01}, x_{00})

and $p_1(x) < p_2(x)$ on (x_{00}, x_{02}). If $p_1(x)$ is nonincreasing and $p_2(x)$ is nondecreasing on $(\min\{x_0, x_{00}\}, \max\{x_0, x_{00}\})$ then $\underset{\sim}{S}(\eta_1, \eta_2)$ is a convex fuzzy set.

Proof: If $x_{01} < x_{00} < x_0 < x_{02}$:

Since $p_1(x)$ is nonincreasing and $p_2(x)$ is nondecreasing on (x_{00}, x_0), $\frac{p_1(x)}{p_2(x)}$ is strictly decreasing from 1. And $\frac{2F_1(x)-1}{1-2F_2(x)}$ is monotone increasing to 1 on $[x_{00}, x_0]$. Hence there exists one and only one point $\bar{x} \in (x_{00}, x_0)$ such that $\frac{p_1(\bar{x})}{p_2(\bar{x})} = \frac{2F_1(\bar{x})-1}{1-2F_2(\bar{x})}$. Since $\frac{p_1(x)}{p_2(x)} \geq \frac{2F_1(x)-1}{1-2F_2(x)}$ on $[x_{00}, \bar{x}]$, $(\mu_{\underset{\sim}{S}(\eta_1, \eta_2)}(x))' \geq 0$, therefore $\mu_{\underset{\sim}{S}(\eta_1, \eta_2)}(x)$ is nondecreasing on $[x_{00}, \bar{x}]$. Similarly, it can be proved that $\mu_{\underset{\sim}{S}(\eta_1, \eta_2)}(x)$ is nonincreasing on $[\bar{x}, x_0]$. Referring to lemma 8, $\underset{\sim}{S}(\eta_1, \eta_2)$ is a convex fuzzy set.

If $x_{01} < x_0 < x_{00} < x_{02}$:

In the same manner as above , it can be known that $\mu_{\underset{\sim}{S}(\eta_1, \eta_2)}(x)$ is a convex fuzzy set.

The whole proof is ended. ∎

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