FUZZY PROXIMITY MATRIX EQUATIONS OF VARIED ORDER

Li Sang Ho

(JiLin teacher's Institute of technician, Changchun, China)

We advance a concept about fuzzy proximity matrix equations of varied order, and give the family of all its solutions.

Keywords: Fuzzy prosimity matrix, Matrix equation of varied order, Greatest solution, Lower solution.

1. Definitions

Definition 1. The matrix

$$R = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \\ r_{n1} & \cdots & r_{nn} \end{bmatrix}$$

is a fuzzy proximity matrix of nth order, if $r_{ii}=1$, $r_{ij}=r_{ji}$, and $r_{ij} \in [0,1]$. The family of all these matrices we denote by $\mathcal{R}$.

Definition 2 Let $X \in \mathcal{R}$ be unknown and $R, S \in \mathcal{R}$ be given. Write $\dot{K} = \{K, K+1, \ldots\}$, where K is a natural number. The equation

$$X^{\dot{k}} \circ R = S \qquad (1)$$

is called the fuzzy proximity matrix equation of varied [illegible]der, and means for every $m \in \dot{K}$ (i.e. $m \geq k$),

$$X^n o R = \underbrace{X o \dots o X}_{m} o R = S.$$

where 'o' denotes the max-min composition. (fuzzy matrix [illegible]).

Definition 3 A is called a n type matrix, if $A^n = A^{n+1}$. Let $\mathcal{A}$ denote the family of all 1 type idempotent matrices. A type n matrix is convergent of index n.

[illegible]. Propositions

Proposition 1 Let $A \in \mathcal{R}$, then

$$A \leq A^2 \leq \dots \leq A^{n-1} = A^n = \dots \quad . \qquad (2)$$

where the $A \leq C$ denotes $a_{ij} \leq c_{ij}$.

Proposition 2 Let

$$\mathcal{X}(y,k) = \{ X \in \mathcal{R} \mid X^{\dot{k}} = y, \ \dot{K} = \{ K, K+1, \dots \} \} \qquad (3)$$

then

$$\mathcal{X}(y,k) \neq \emptyset \quad \text{iff} \quad y = y^2.$$

proof. If $y = y^2$, then

$$y = y^2 = \dots = y^k = y^{k+1} = \dots$$

therefore $y \in \mathcal{X}(y,k)$, i.e. $\mathcal{X}(y,k) \neq \emptyset$.

conversely, if $\mathcal{X}(y,k) \neq \emptyset$, assume $A \in \mathcal{X}(y,k)$, then

$$A^k = A^{k+1} = \dots = y,$$

and $y^2 = A^{2k}$.

but $A^k = A^{2k}$,

thereupon $y = y^2$

Proposition 3 Let

$$\mathcal{X}(R,S,\dot{K}) = \{ X \in \mathcal{R} \mid X^{\dot{k}} o R = S, \ \dot{K} = \{ K, K+1, \dots \} \} \qquad (4)$$

and

$$\mathcal{Y}(R,S) = \{ Y \in \mathcal{A} \mid Y o R = S \} \qquad (5)$$

then $\mathcal{X}(R,S,\dot{K})\neq\emptyset$ iff $\mathcal{Y}(R,S)\neq\emptyset$

Proof. If $\mathcal{X}(R,S,\dot{K})\neq\emptyset$, let $A\in\mathcal{X}(R,S,\dot{K})$, then $A^{\dot{k}}oR=S$.

But, $A^k=A^{k+1}=\dots=A^{2k}=\dots$.

This shows $A^k=(A^k)^2$.

Therefore $A^k\in\mathcal{Y}(R,S)$, i.e. $\mathcal{Y}(R,S)\neq\emptyset$.

Conversely, if $\mathcal{Y}(R,S)\neq\emptyset$, then there is a B

$B\in\mathcal{Y}(R,S)$, So that

$BoR = S$,

and $B = B^2$

By $B=B^2$, we obtained that

$B=B^2=\dots=B^k=B^{k+1}=\dots$

According to $BoR=S$, we have

$B^{\dot{k}}oR = S$

Thereupon $B\in\mathcal{X}(R,S,K)$, i.e.

$\mathcal{X}(R,S,K)\neq\emptyset$.

Proposition 4 Let

$$\mathcal{X}(y,m)=\{X\in\mathcal{R}\mid X^m=y,\ y\in\mathcal{A}\} \qquad (6)$$

and

$$\mathcal{X}(y,k)=\{X\in\mathcal{R}\mid X^k=y,\ y\in\mathcal{A}\} \qquad (7)$$

then for $m\geq k$, we have

$$\mathcal{X}(y,m)\supseteq\mathcal{X}(y,k) \qquad (8)$$

Proof. For any $X_o\in\mathcal{X}(y,k)$, then

$$X_o^k = y$$

and

$$y = y^2 \qquad (9)$$

Because $m\geq k$, there exists a positive integer α , such that $\alpha k\geq m$.

and then, by (2) and (10), we obtain that

$$X_0^k \subseteq X_0^m \subseteq X_0^{\alpha k},$$

and

$$X_0^{\alpha k} = y^{\alpha} = y = X_0^k$$

Therefore

$$X_0^m = X_0^k = y, \quad \text{i.e.}$$

$$X_0 \in X(y,m).$$

<u>Proposition 5</u>

$$\mathscr{X}(R,S,\dot{K}) = \bigcup_{y \in \mathscr{Y}(S,S)} (y,k) \qquad (10)$$

Where R be not the universal matrix and $K>1$.

Proof. We can easily obtain that

$$\mathscr{X}(R,S,\dot{K}) = \bigcap_{m \geq k} \Big(\bigcup_{y \in \mathscr{Y}(R,S)} \mathscr{X}(y,m)$$

By (8), obtain that

$$\mathscr{X}(y,m) \supseteq \mathscr{X}(y,k),$$

for $m \geq k$.

Therefore

$$\bigcup_{y \in \mathscr{Y}(R,S)} \mathscr{X}(y,m) \supseteq \bigcup_{y \in \mathscr{Y}(R,S)} \mathscr{X}(y,k)$$

for $m \geq k$.

Thereupon

$$\bigcap_{m \geq k} \Big(\bigcup_{y \in \mathscr{Y}(R,S)} \mathscr{X}(y,m) \Big) = \bigcup_{y \in \mathscr{Y}(R,S)} \mathscr{X}(y,k)$$

i.e.

$$\mathscr{X}(R,S,\dot{K}) = \bigcup_{y \in \mathscr{Y}(R,S)} \mathscr{X}(y,k) .$$

In particular

$$\mathcal{X}(y,\dot{k}) = \bigcup_{y\in\mathcal{A}} \mathcal{X}(y,k). \qquad (11)$$

Proposition 6 Let $A = (a_{ij}) \in \mathcal{R}$,

$B=(b_{ij})\in \mathcal{R}$ and $AoB=C=(C_{ij})$, then

$$C \supseteq A \qquad (12)$$

and

$$C \supseteq B \qquad (13)$$

Proof. In fact, for any i,j=1, ... , n, we have

$$c_{ij} = \max_{1\le k\le n} \min (a_{ik}, b_{ki})$$

$$= \max \{ \min(a_{ij}, b_{jj}), \min(a_{ii}, b_{ij}), \max_{k\neq i,j} \min(a_{ik}, b_{kj})\}$$

$$= \max \{a_{ij}, b_{ij}, \max_{k\neq i,j} \min(a_{ik}, b_{kj}) \}$$

$$\geq \max(a_{ij}, b_{ij}).$$

Therefore $C \supseteq A$ and $C \supseteq B$

Proposition 7 If R=S or $R\leq S$, $S=S^2$, then

$$\mathcal{Y}(R,S) \neq \emptyset$$

Proof. First suppose that R=S, then IoR=S and

$$I^2 = I = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

Therefore $I \in \mathcal{Y}(R,S)$.

If $R \leq S$ and $S=S^2$, then

$$S \circ R \subseteq S \circ S = S.$$

By (12), obtain that

$$S \circ R \supseteq S.$$

Therefore

$$S \circ R = S \quad \text{and} \quad S = S^2 ,$$

i.e. $S \in \mathcal{Y}(R,S)$.

3. Examples

Example 1 Let us consider equation (1) with

$$R = \begin{bmatrix} 1 & 0.6 & 0.3 & 0 \\ 0.6 & 1 & 0.1 & 0.2 \\ 0.3 & 0.1 & 1 & 0.7 \\ 0 & 0.2 & 0.7 & 1 \end{bmatrix},$$

$$S = \begin{bmatrix} 1 & 0.6 & 0.4 & 0.4 \\ 0.6 & 1 & 0.4 & 0.4 \\ 0.4 & 0.4 & 1 & 0.7 \\ 0.4 & 0.4 & 0.7 & 1 \end{bmatrix}$$

$$\dot{K} = \{2, 3, 4, \ldots\}$$

According to the paper (2) we obtain that

$$\begin{bmatrix} 1 & 0 & 0 & 0.4 \\ 0 & 1 & 0.4 & 0 \\ 0 & 0.4 & 1 & 0 \\ 0.4 & 0 & 0 & 1 \end{bmatrix} \subseteq X^2 \subseteq \begin{bmatrix} 1 & 0.6 & 0.4 & 0.4 \\ 0.6 & 1 & 0.4 & 0.4 \\ 0.4 & 0.4 & 1 & 0.7 \\ 0.4 & 0.4 & 0.7 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0.4 & 0 \\ 0 & 1 & 0 & 0.4 \\ 0.4 & 0 & 1 & 0 \\ 0 & 0.4 & 0 & 1 \end{bmatrix} \subseteq X^2 \subseteq \begin{bmatrix} 1 & 0.6 & 0.4 & 0.4 \\ 0.6 & 1 & 0.4 & 0.4 \\ 0.4 & 0.4 & 1 & 0.7 \\ 0.4 & 0.4 & 0.7 & 1 \end{bmatrix}$$

we obtain that

$$\begin{bmatrix} 1 & 0 & 0 & 0.4 \\ 0 & 1 & 0.4 & 0 \\ 0 & 0.4 & 1 & 0 \\ 0.4 & 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 & 0.4 \\ 0 & 1 & 0.4 & 0 \\ 0 & 0.4 & 1 & 0 \\ 0.4 & 0 & 0 & 1 \end{bmatrix},$$

and so

$$\begin{bmatrix} 1 & 0 & 0.4 & 0 \\ 0 & 1 & 0 & 0.4 \\ 0.4 & 0 & 1 & 0 \\ 0 & 0.4 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0.4 & 0 \\ 0 & 1 & 0 & 0.4 \\ 0.4 & 0 & 1 & 0. \\ 0 & 0.4 & 0 & 1 \end{bmatrix}$$

[illegible]

$$\begin{bmatrix} 1 & 0.6 & 0.4 & 0.4 \\ 0.6 & 1 & 0.4 & 0.4 \\ 0.4 & 0.4 & 1 & 0.7 \\ 0.4 & 0.4 & 0.7 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0.6 & 0.4 & 0.4 \\ 0.6 & 1 & 0.4 & 0.4 \\ 0.4 & 0.4 & 1 & 0.7 \\ 0.4 & 0.4 & 0.7 & 1 \end{bmatrix}$$

According to the paper (3), the solutions of the equations

$$X^2 = \begin{bmatrix} 1 & 0 & 0 & 0.4 \\ 0 & 1 & 0.4 & 0 \\ 0 & 0.4 & 1 & 0 \\ 0.4 & 0 & 0 & 1 \end{bmatrix}$$

and

$$X^2 = \begin{bmatrix} 1 & 0 & 0.4 & 0 \\ 0 & 1 & 0 & 0.4 \\ 0.4 & 0 & 1 & 0 \\ 0 & 0.4 & 0 & 1 \end{bmatrix}$$

are respectively

$$X = \begin{bmatrix} 1 & 0 & 0 & 0.4 \\ 0 & 1 & 0.4 & 0 \\ 0 & 0.4 & 1 & 0 \\ 0.4 & 0 & 0 & 1 \end{bmatrix}$$

or

$$X = \begin{bmatrix} 1 & 0 & 0.4 & 0 \\ 0 & 1 & 0 & 0.4 \\ 0.4 & 0 & 1 & 0 \\ 0 & 0.4 & 0 & 1 \end{bmatrix}$$

The greatest solution of the equation

$$X^2 = \begin{bmatrix} 1 & 0.6 & 0.4 & 0.4 \\ 0.6 & 1 & 0.4 & 0.4 \\ 0.4 & 0.4 & 1 & 0.7 \\ 0.4 & 0.4 & 0.7 & 1 \end{bmatrix}$$

is

$$X = \begin{bmatrix} 1 & 0.6 & 0.4 & 0.4 \\ 0.6 & 1 & 0.4 & 0.4 \\ 0.4 & 0.4 & 1 & 0.7 \\ 0.4 & 0.4 & 0.7 & 1 \end{bmatrix}$$

Therefore, by (10) we obtain

$$\mathcal{X}(R,S,\dot{K}) = \left\{ X \;\middle|\; \begin{bmatrix} 1 & 0 & 0 & 0.4 \\ 0 & 1 & 0.4 & 0 \\ 0 & 0.4 & 1 & 0 \\ 0.4 & 0 & 0 & 1 \end{bmatrix} \subseteq X \subseteq \begin{bmatrix} 1 & 0.6 & 0.4 & 0.4 \\ 0.6 & 1 & 0.4 & 0.4 \\ 0.4 & 0.4 & 1 & 0.7 \\ 0.4 & 0.4 & 0.7 & 1 \end{bmatrix} \right\}$$

$$\cup \left\{ X \;\middle|\; \begin{bmatrix} 1 & 0 & 0.4 & 0 \\ 0 & 1 & 0 & 0.4 \\ 0.4 & 0 & 1 & 0 \\ 0 & 0.4 & 0 & 1 \end{bmatrix} \subseteq X \subseteq \begin{bmatrix} 1 & 0.6 & 0.4 & 0.4 \\ 0.6 & 1 & 0.4 & 0.4 \\ 0.4 & 0.4 & 1 & 0.7 \\ 0.4 & 0.4 & 0.7 & 1 \end{bmatrix} \right\}$$

Example 2 Let us consider equation

$$X^4 = \begin{bmatrix} 1 & 0.4 & 0.8 & 0.5 & 0.5 \\ 0.4 & 1 & 0.4 & 0.4 & 0.4 \\ 0.8 & 0.4 & 1 & 0.5 & 0.5 \\ 0.5 & 0.4 & 0.5 & 1 & 0.6 \\ 0.5 & 0.4 & 0.5 & 0.6 & 1 \end{bmatrix}$$

According to the paper (3) we obtain that the greatest solution of the equation

$$X^4 = \begin{bmatrix} 1 & 0.4 & 0.8 & 0.5 & 0.5 \\ 0.4 & 1 & 0.4 & 0.4 & 0.4 \\ 0.8 & 0.4 & 1 & 0.5 & 0.5 \\ 0.5 & 0.4 & 0.5 & 1 & 0.6 \\ 0.5 & 0.4 & 0.5 & 0.6 & 1 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0.4 & 0.8 & 0.5 & 0.5 \\ 0.4 & 1 & 0.4 & 0.4 & 0.4 \\ 0.8 & 0.4 & 1 & 0.5 & 0.5 \\ 0.5 & 0.4 & 0.5 & 1 & 0.6 \\ 0.5 & 0.4 & 0.5 & 0.6 & 1 \end{bmatrix}$$

and the lower solutions of this equation be of form

$$\mathcal{X}_{ij}(y,k) = \begin{bmatrix} 1 & a_1 & 0.8 & b_1 & b_2 \\ a_1 & 1 & a_2 & a_3 & a_4 \\ 0.8 & a_2 & 1 & b_3 & b_4 \\ b_1 & a_3 & b_3 & 1 & 0.6 \\ b_2 & a_4 & b_4 & 0.6 & 1 \end{bmatrix}$$

Where $a_i=0$, $b_j=0$ and $a_m=0.4$, $b_n=0.5$, for $i,j \in \{1, 2, 3, 4\}$, $i \neq m$, $j \neq n$.

By (12) we obtain

$$S(R, \dot{k}) = \bigcup_{1 \leq i,\, j \leq 4} \left\{ X \;\middle|\; \begin{bmatrix} 1 & a_1 & 0.8 & b_1 & b_2 \\ a_1 & 1 & a_2 & a_3 & a_4 \\ 0.8 & a_2 & 1 & b_3 & b_4 \\ b_1 & a_3 & b_3 & 1 & 0.6 \\ b_2 & a_4 & b_4 & 0.6 & 1 \end{bmatrix} \subseteq X \subseteq \begin{bmatrix} 1 & 0.4 & 0.8 & 0.5 & 0.5 \\ 0.4 & 1 & 0.4 & 0.4 & 0.4 \\ 0.8 & 0.4 & 1 & 0.5 & 0.5 \\ 0.5 & 0.4 & 0.5 & 1 & 0.6 \\ 0.5 & 0.4 & 0.5 & 0.6 & 1 \end{bmatrix} \right\}$$

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