A NOTE ON THE INVERSES OF FUZZY RELATIONS

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Let X, Y be two universes of discourse and let A,B be fuzzy relations in $X \times Y$ and $Y \times X$, respectively, viz. $A : X \times Y \to [0,1]$, $B : Y \times X \to [0,1]$.

Definition. B is called right, left or central inverse of A, if it fulfills one of the following conditions:

(R) $A^T * A * B = A^T$,

(L) $B * A * A^T = A^T$,

(C) $A * B * A = A$,

respectively, where "*" denotes the sup-* composition of fuzzy relations with given monotonic commutative binary operation * in [0,1] (cf. [2], [3]).

(R), (L), (C) may be also rewritten in an equivalent form:

(R) $\sup_{p \in X, s \in Y} (A(q,p) * A(p,s) * B(s,r)) = A(q,r)$ for $r \in X$, $q \in Y$,

(L) $\sup_{r \in X, p \in Y} (B(s,r) * A(r,p) * A(q,p)) = A(q,s)$ for $q \in X$, $s \in Y$,

(C) $\sup_{s \in X, q \in Y} (A(p,q) * B(q,s) * A(s,r)) = A(P,r)$ for $p \in X$, $r \in Y$.

The existence of the two first inverses can be immediately characterized by using theory of the fuzzy relational equations (cf. [2], [3], [5]-[7]). Let "Δ" denote an inverse operator with respect to sup-* composition defined as (cf. [3])

$$(A \Delta B)(p,q) = \inf_{r \in Y} (A(p,r) \to B(r,q)) \quad \text{for } p, q \in X,$$

where

$$a \to b = \max\{c \in [0,1] \mid a * c \leqslant b\}$$

if this maximum exists for any $a, b \in [0,1]$. (Note that any t-norm (cf. [3], [5]),

$$a \; t \; b = t^{-1}(t(a)t(b)) \quad \text{for } a, b \in [0,1]$$

with arbitrary isotone bijection $t : [0,1] \to [0,1]$ satisfies the above assumption on operation * and we have

$$a \to b = t^{-1}(\min(1, \frac{t(b)}{t(a)})) \quad \text{for } a, b \in [0,1] \quad).$$

Proposition 1. A has the right inverse iff $B = (A^T * A) \Delta A^T$ fulfils (R). Moreover, this formula gives the greatest right inverse (in lattice order of fuzzy relations) if the set of inverses is nonempty.

Proposition 2. A has the left inverse iff $B = ((A^T * A) \Delta A)^T$ fulfils (L). When this is the case, it is the greatest left inverse.

In order to find the central inverse of A let us rewrite (C) in a slightly modified manner

(C') $C \circ B = A$, i.e. $\sup_{(s,q)\in Y\times X} (C(p,r,s,q) * B(s,q)) = A(p,r)$,

where $C : X \times Y \times X \times Y \to [0,1]$,

$C(p,r,s,q) = A(p,s) * A(q,r)$ for $p, q \in X$, $r, s \in Y$.

Now the fuzzy relational equation theory yields the following

Proposition 3. A has the central inverse iff $B = C^T \vartriangle A$ fulfils (C'), where

$$B(s,q) = \inf_{(p,r)\in X\times Y} (C(p,r,s,q) \to A(p,r)) \quad \text{for} \quad (s,q) \in Y \times X$$

is the greatest element in the set of central inverses.

The determination of the central inverse has been already discussed in [4] but in a quite different manner (for * = min).

In the case of finite X and Y, the entire family of the above treated inverses can be determined by the method introduced in [1].

REFERENCES

[1] E. Czogała, J. Drewniak, W. Pedrycz, Fuzzy relation equations on a finite set, Fuzzy Set and Systems 7(1982) 89-101.

[2] J. Drewniak, Note on fuzzy relation equations, BUSEFAL 12(1982) 50-51.

[3] J. Drewniak, Fuzzy relation equations and inequalities (to appear).

[4] Ch.-Z. Luo, Generalized inverses of fuzzy matrix, in: Approximate Reasoning in Decision Analysis, M.M. Gupta, E. Sanchez (eds.), North-Holland, Amsterdam 1982, 57-60.

[5] W. Pedrycz, Fuzzy relational equations with triangular norms and their resolutions, BUSEFAL 11(1982) 24-32.

[6] W. Pedrycz, Fuzzy relational equations with generalized connectives and their applications, Fuzzy Sets and Systems 10(1983) 185-202.

[7] E. Sanchez, Resolution of composite fuzzy relation equation, Inform. and Control 30(1967) 38-48.