On the (N)Fuzzy Integral
(Abstract)

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1 INTRODUCTION

Wang [4,5,6,7,8] studied some structural characteristics of fuzzy measure; and used them in the theory of convergences of sequence of measurable functions and of (S)fuzzy integrals, obtained a series of new results.

Zhao [9] introduced the concept of (N)fuzzy integral and studied some properties of (N)fuzzy integral and of sequence of (N)fuzzy integrals.

In this paper, we shall further discuss the (N)fuzzy integral by using the structural characteristics of fuzzy measure proposed in [4,5,6,7,8] .

2 BASIC DEFINITIONS

Let X be a nonempty set, $\mathcal{F}$ be a σ-algebra of subsets of X, $\mu : \mathcal{F} \longrightarrow [0,\infty]$ be a fuzzy measure on $(X, \mathcal{F})$.

The following definitions are introduced from [4,7] .

Definition 2.1 The fuzzy measure μ is called subadditive if for every $A \in \mathcal{F}$, and every $B \in \mathcal{F}$, we have

$$\mu(A \cup B) \leq \mu(A) + \mu(B).$$

Definition 2.2 The fuzzy measure μ is called null-additive, denoted by 0-add., if we have $\mu(A \cup B) = \mu(B)$, whenever $A \in \mathcal{F}$, $B \in \mathcal{F}$, $\mu(A) = 0$.

Definition 2.3 Let $A \in \mathcal{F}$, $\mu(A) < \infty$. μ is called pseudo-null-additive with respect to A, denoted by p. 0-add./A, if we have $\mu(B \cup C) = \mu(C)$, whenever $B \in A \cap \mathcal{F}$, $C \in A \cap \mathcal{F}$, $\mu(A-B) = \mu(A)$.

Definition 2.4 μ is called autocontinuous from above, denoted by autoc.$\downarrow$ (resp. autocontinuous from below, denoted by autoc.$\uparrow$) if $\forall A \in \mathcal{F}$, $\forall \{B_n\} \subset \mathcal{F}$,

$$\mu(B_n) \longrightarrow 0 \Longrightarrow \mu(A \cup B_n) \longrightarrow \mu(A)$$

$$(\text{resp. } \mu(A-B_n) \longrightarrow \mu(A));$$

μ is called autocontinuous, denoted by autoc., if it is both autoc.↓and autoc.↑.

Definition 2.5 Let $A \in \mathcal{F}$, $\mu(A) < \infty$. μ is called pseudo-autocontinuous from above with respect to A, denoted by p. autoc.↓/A (resp. pseudo-autocontinuous from below with respect to A, denoted by p. autoc.↑/A) if

$$\mu(B_n \cap A) \longrightarrow \mu(A) \Longrightarrow \mu((A-B_n) \cup C) \longrightarrow \mu(C)$$
$$(\text{resp. } \mu(B_n \cap C) \longrightarrow \mu(C)),$$

for every $C \in A \cap \mathcal{F}$; μ is called pseudo-autocontinuous with respect to A, if it is both p. autoc.↓/A and p. autoc.↑/A.

Definition 2.6 A class $\mathcal{C}$ of sets in $\mathcal{F}$ is called a chain if whenever $C_1 \in \mathcal{C}$, $C_2 \in \mathcal{C}$, then either $C_1 \subset C_2$ or $C_2 \subset C_1$. A chain $\mathcal{C}$ is called μ-bounded, if there exists $M > 0$, such that $\mu(C) \leq M$ for $\forall C \in \mathcal{C}$.

Definition 2.7 μ is called local- uniformly autocontinuous from above (resp. local-uniformly autocontinuous from below), if it is autoc.↓, and for every μ-bounded chain $\mathcal{C} \subset \mathcal{F}$ and every $\varepsilon > 0$, there exists a $\delta = \delta(\mathcal{C}, \varepsilon) > 0$, such that

$$\mu(A \cup B) \leq \mu(A) + \varepsilon ,$$
$$(\text{resp. } \mu(A-B) \geq \mu(A) - \varepsilon),$$

whenever $A \in \mathcal{C}$, $B \in \mathcal{F}$, $\mu(B) < \delta$; μ is called local-uniformly autocontinuous if it is both local-uniformly autocontinuous from above and local-uniformly autocontinuous from below.

Wang [4,7] gave the following results:

Theorem 2.1 Let μ be a fuzzy measure, then

(1) μ is subadditive.$\Longrightarrow \mu$ is autoc..

(2) μ is autoc.↓.(resp. autoc.↑.)$\Longrightarrow \mu$ is 0-add..

(3) μ is p. autoc.↓/A . (resp. p. autoc.↑/A)$\Longrightarrow \mu$ is p. 0-add./A .

(4) μ is autoc.↓.$\Longleftrightarrow \mu$ is local-uniformly autocontinuos from above.

(5) μ is autoc.↑.$\Longleftrightarrow \mu$ is local-uniformly autocontinuous from below.

Now we shall introduce the concept of local-uniform pseudo-autocontinuity, and give the relation between this concept and the pseudo-autocontinuity.

Definition 2.8 Let $A\in\mathcal{F}$, $\mu(A)<\infty$. μ is called local-uniformly pseudo-autocontinuous from above with respect to A, (resp. local-uniformly pseudo-autocontinuous from below with respect to A,) if for every chain $\mathcal{C}\in A\cap\mathcal{F}$ and every $\varepsilon>0$, there exists a $\delta=\delta(\mathcal{C},\varepsilon)>0$, such that

$$\mu((A-B)\cup C) - \mu(C) < \varepsilon$$

$$(\text{resp. } \mu(C) - \mu(B\cap C)<\varepsilon);$$

whenever $C\in\mathcal{C}$, $B\in\mathcal{F}$ and $\mu(A) - \mu(A\cap B)<\delta$; μ is called local-uniformly pseudo-autocontinuous with respect to A, if it is both local-uniformly pseudo-autocontinuous from above with respect to A and local-uniformly pseudo-autocontinuous from below with respect to A.

Theorem 2.2 Let $A\in\mathcal{F}$, $\mu(A)<\infty$. Then

(1) μ is p. autoc.$\downarrow$/A.$\Longleftrightarrow\mu$ is local-uniformly pseudo-autocontinuous from above with respect to A.

(2) μ is p. autoc.$\uparrow$/A.$\Longleftrightarrow\mu$ is local-uniformly pseudo-autocontinuous from below with respect to A.

3 THE NECESSARY AND SUFFICIENT CONDITION FOR THE (N)FUZZY INTEGRABILITY

The discussions in this section and in the following sections will be done on a given fuzzy measure space $(X,\mathcal{F},\mu)$, the functions used will be non negative measurable functions and these functions will be denoted by f, f_n, $n=1,2,\cdots$.

In this paper, we shall make the following conventions: for every $\alpha>0$,

$$F_\alpha = \{f \geq \alpha\}, \qquad F_{\bar\alpha} = \{f > \alpha\},$$

$$F^n_\alpha = \{f_n \geq \alpha\}, \qquad F^n_{\bar\alpha} = \{f_n > \alpha\},$$

$n=1,2,\cdots$.

Definition 3.1 The (N)fuzzy integral of a function f over $A \in \mathcal{F}$ is defined as

$$(N)\int_A f d\mu = \sup_{\alpha>0} [\alpha \cdot \mu(F_{\bar\alpha} \cap A)].$$

Definition 3.2 Let $A \in \mathcal{F}$. f is called (N)fuzzy integrable over A, denoted by $f \in L_A(\mu)$ if

$$(N)\int_A f d\mu < \infty.$$

Zhao [9] gave the following sufficient condition for the (N)fuzzy integrability.

Theorem 3.1 If $\mu(X) < \infty$ and f is bounded, then for every $A \in \mathcal{F}$, $f \in L_A(\mu)$.

Now we shall give a necessary and sufficient condition for the (N)fuzzy integrability.

Theorem 3.2 Let $A \in \mathcal{F}$, then the following two conditions are equivalent:

(1) $f \in L_A(\mu)$;

(2) $\forall \alpha > 0$, $\mu(F_{\bar\alpha} \cap A) < \infty$ and

$$\overline{\lim_{\alpha \to \infty}} \alpha \cdot \mu(F_{\bar\alpha} \cap A) < \infty,$$

$$\overline{\lim_{\alpha \to 0}} \alpha \cdot \mu(F_{\bar\alpha} \cap A) < \infty.$$

4 SEVERAL EQUIVALENT DEFINITIONS OF (N)FUZZY INTEGRAL

Theorem 4.1

$$(N)\int_A f d\mu = \sup_{\alpha>0} [\alpha \cdot \mu(F_{\bar\alpha} \cap A)] = \sup_{\alpha>0} [\alpha \cdot \mu(F_\alpha \cap A)].$$

Corollary: Let $A \in \mathcal{F}$. The following two conditions are equivalent:

(1) $f \in L_A(\mu)$;

(2) $\forall \alpha > 0$, $\mu(F_\alpha \cap A) < \infty$ and

$$\overline{\lim_{\alpha \to \infty}} \alpha \cdot \mu(F_\alpha \cap A) < \infty,$$

$$\overline{\lim_{\alpha \to 0}} \alpha \cdot \mu(F_\alpha \cap A) < \infty.$$

Theorem 4.2 $(N)\int_A f d\mu = \sup_{E \in \mathcal{F}} [(\inf_{x \in E} f(x)) \cdot \mu(A \cap E)]$

$$= \sup_{E \in \beta(f)} [(\inf_{x \in E} f(x)) \cdot \mu(A \cap E)]$$

where $\beta(f)$ is the σ-algebra generated by f. (since f is measurable, $\beta(f)$ is subset of $\mathcal{F}$.)

Now we shall give another equivalent definition of (N) fuzzy integral.

For every non negative simple function

$$s = \sum_{i=1}^{n} \alpha_i \cdot \chi_{A_i}$$

(where $A_i \in \mathcal{F}$, $i=1,2,\cdots,n$, $\alpha_i \neq \alpha_j$ $(i \neq j)$,) we define

$$Q_A(s) = \bigvee_{i=1}^{n} [\alpha_i \cdot \mu(A \cap A_i)] \qquad \forall A \in \mathcal{F}.$$

Theorem 4.3 $\quad (N)\int_A f d\mu = \sup_{0 \leq s \leq f} Q_A(s).$

5 PROPERTIES OF (N)FUZZY INTEGRAL

Shao [9] gave the following results:

Theorem 5.1 Let $A, B \in \mathcal{F}$, then

(1) $\mu(A) = 0 \Longrightarrow (N)\int_A f d\mu = 0;$

(2) $(N)\int_A f d\mu = 0 \Longrightarrow \mu(\{f > 0\} \cap A) = 0;$

(3) $f_1 \leq f_2 \Longrightarrow (N)\int_A f_1 d\mu \leq (N)\int_A f_2 d\mu$;

(4) $(N)\int_A f d\mu = (N)\int f \cdot \chi_A d\mu$, where χ_A is the characteristic function of A;

(5) $A \subset B \Longrightarrow (N)\int_A f d\mu \leq (N)\int_B f d\mu$;

(6) $(N)\int_A a d\mu = a \cdot \mu(A), \quad \forall a > 0;$

(7) $(N)\int_A (f_1 \vee f_2) d\mu \geq ((N)\int_A f_1 d\mu) \vee ((N)\int_A f_2 d\mu);$

(8) $(N)\int_A (f_1 \wedge f_2) d\mu \leq ((N)\int_A f_1 d\mu) \wedge ((N)\int_A f_2 d\mu);$

(9) $(N)\int_{A \cup B} f d\mu \geq ((N)\int_A f d\mu) \vee ((N)\int_B f d\mu);$

(10) $(N)\int_{A\cap B} f d\mu \le ((N)\int_A f d\mu) \wedge ((N)\int_B f d\mu)$;

(11) $(N)\int_A cf d\mu = c(N)\int_A f d\mu, \quad \forall c > 0$;

(12) $(N)\int_A (c \vee f) d\mu = (c\cdot\mu(A)) \vee (N)\int_A f d\mu, \quad \forall c > 0$.

We can prove the following theorems:

Theorem 5.2 Let $A \in \mathcal{F}$, $a > 0$, then

$$(N)\int_A (f+a) d\mu \le (N)\int_A f d\mu + a\mu(A).$$

Definition 5.1 We say f_1 and f_2 are equal almost everywhere, denoted by $f_1 = f_2$ a.e., if

$$\mu(\{f_1 \ne f_2\}) = 0.$$

Definition 5.2 Let $A \in \mathcal{F}$, $\mu(A) < \infty$. We say f_1 and f_2 are equal pseudo-almost everywhere on A, denoted by $f_1 = f_2$ **p.** a.e. on A, if

$$\mu(\{f_1 = f_2\} \cap A) = \mu(A).$$

Theorem 5.3 Let $A \in \mathcal{F}$, $\mu(A) < \infty$, $a > 0$. If $|f_1 - f_2| \le a$ on A, then

$$\left|(N)\int_A f_1 d\mu - (N)\int_A f_2 d\mu\right| \le a\mu(A).$$

Theorem 5.4 Whenever $f_1 = f_2$ a.e., it holds

$$(N)\int f_1 d\mu = (N)\int f_2 d\mu,$$

if and only if μ is 0-add..

Theorem 5.5 Let $A \in \mathcal{F}$, $\mu(A) < \infty$. Whenever $f_1 = f_2$ p.a.e. on A, it holds

$$(N)\int_A f_1 d\mu = (N)\int_A f_2 d\mu$$

if and only if μ is p. 0- add./A

6 THE TRANSFORMATION THEOREMS

Zhao [9] proved the following theorem only for the case of $f \in L_A(\mu)$, and besides, the proof given in [9] is long. In fact, we can prove it more briefly.

<u>Theorem</u> 6.1 $(N)\int_A f d\mu = (N)\int_0^{\infty} \mu(F_{\bar{\alpha}} \cap A) dm$

$$= (N)\int_0^{\infty} \mu(F_{\alpha} \cap A) dm$$

where m is the Lebesgue measure on $[0,\infty]$.

We can also give a transformation theorem for the Lebesgue integral.

<u>Theorem</u> 6.2 Let $(X,\mathfrak{F},\mu)$ be a measure space, f be a measurable function, then

$$(L)\int_A f d\mu = (L)\int_0^{\infty} \mu(F_{\bar{\alpha}} \cap A) d\alpha$$

$$= (L)\int_0^{\infty} \mu(F_{\alpha} \cap A) d\alpha .$$

for every $A \in \mathfrak{F}$.

From this theorem we know that Theorem 4.10 in [9] is a direct conclusion of Theorem 3.8 in [9], hence we can omit the long and boring proof for Theorem 4.10 given in [9].

7 CONVERGENCE THEOREMS OF SEQUENCE OF (N)FUZZY INTEGRALS

<u>Theorem</u> 7.1 Let $A \in \mathfrak{F}$, $\mu(A) < \infty$. If $\{f_n\}$ converges to f uniformly on A then

$$\lim_{n\to\infty}(N)\int_A f_n d\mu = (N)\int_A f d\mu .$$

<u>Definition</u> 7.1 We say $\{f_n\}$ converges to f in measure, denoted by $f_n \xrightarrow{\mu} f$, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty}\mu(\{|f_n - f| \geqslant \varepsilon\}) = 0.$$

<u>Definition</u> 7.2 Let $A \in \mathfrak{F}$, $\mu(A) < \infty$. We say $\{f_n\}$ converges to f pseudo-in measure on A, denoted by $f_n \xrightarrow{P.\mu} f$ on A, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty}\mu(\{|f_n-f|<\varepsilon\}\cap A) = \mu(A).$$

Theorem 7.2 (1) Let $\mu(X)<\infty$. Whenever $f\in\mathbb{F}$, $\{f_n\}\subset\mathbb{F}$ is uniformly bounded and $f_n\xrightarrow{\mu}f$, then

$$\lim_{n\to\infty}(N)\int f_n d\mu = (N)\int f d\mu ,$$

if and only if μ is autoc..

(2) Let $A\in\mathcal{F}$, $\mu(A)<\infty$. Whenever $f\in\mathbb{F}$, $\{f_n\}\subset\mathbb{F}$ is uniformly bounded and $f_n\xrightarrow{p.\mu}f$ on A, then

$$\lim_{n\to\infty}(N)\int_A f_n d\mu = (N)\int_A f d\mu ,$$

if and only if μ is p. autoc./A.

Theorem 7.3 (1) Let $\mu(X)<\infty$. Whenever $f\in\mathbb{F}$, $\{f_n\}\subset\mathbb{F}$, $f_n\leq f$ $(n=1,2,\cdots)$ and $f_n\xrightarrow{\mu}f$, then

$$\lim_{n\to\infty}(N)\int f_n d\mu = (N)\int f d\mu ,$$

if and only if μ is autoc.↑.

(2) Let $A\in\mathcal{F}$, $\mu(A)<\infty$. Whenever $f\in\mathbb{F}$, $\{f_n\}\subset\mathbb{F}$, $f_n\leq f$ $(n=1,2,\cdots)$ and $f_n\xrightarrow{p.\mu}f$ on A, then

$$\lim_{n\to\infty}(N)\int_A f_n d\mu = (N)\int_A f d\mu ,$$

if and only if μ is p. autoc.↑/A.

Theorem 7.4 Let $\mu(X)<\infty$, $\{f_n\}$ be a uniformly bounded sequence, if $f_n\longrightarrow f$ then

$$\lim_{n\to\infty}(N)\int f_n d\mu = (N)\int f d\mu .$$

Definition 7.3 We say that $\{f_n\}$ converges to f almost everywhere, denoted by $f_n\longrightarrow f$ a.e., if

$$\mu(\{\lim_{n\to\infty} f_n = f \text{ is not true}\}) = 0.$$

Definition 7.4 Let $A\in\mathcal{F}$, $\mu(A)<\infty$. We say that $\{f_n\}$

converges to f pseudo-almost everywhere on A, denoted by $f_n \longrightarrow f$ p.a.e. on A, if

$$\mu(\{\lim_{n\to\infty} f_n = f\} \cap A) = \mu(A).$$

Theorem 7.5 (1) Let $\mu(X) < \infty$. Whenever $f \in \mathbb{F}$, $\{f_n\} \subset \mathbb{F}$ is uniformly bounded on X and $f_n \longrightarrow f$ a.e., then

$$\lim_{n\to\infty}(N)\int f_n d\mu = (N)\int f d\mu ,$$

if and only if μ is 0-add..

(2) Let $A \in \mathfrak{F}$, $\mu(A) < \infty$. Whenever $f \in \mathbb{F}$, $\{f_n\} \subset \mathbb{F}$ is uniformly bounded on A and $f_n \longrightarrow f$ p.a.e. on A, then

$$\lim_{n\to\infty}(N)\int_A f_n d\mu = (N)\int_A f d\mu ,$$

if and only if μ is p. 0-add./A.

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