TOWARDS A NEW GENERATION OF FUZZY SETS

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1. Towards an extended many-valued logic

The concept of fuzzy sets is based, in principle, on the Łukasiewicz's many-valued continuous logic (see e.g./5/,/15/). While that logic is an extension of the idea of the three-valued one (see /9/). What is the essence of logical systems with three values? Łukasiewicz wrote in /8/:

> "Three-valued logic is a system of non-Aristotelian logic, since it assumes that in addition to true and false propositions there also are propositions that are neither true nor false, and hence, that there exists a third logical value. That third logical value may be interpreted as possibility and may be symbolized by 2".

In later papers Łukasiewicz denoted that third logical value always by 1/2. However he was not sure how to interprete the 1/2. He gave, in the author's opinion, only one of possible interpretations (cf. e.g. the paper /14/, where Smolka suggested that the third value deals with propositional formulae in which the variables are not specified). In that paper we shall attempt to give another, more general, interpretation.

The considerations presented below are motivated by the semimonograph /11/ "Nonsense-logic systems" written by M.Piróg-Rzepecka. Roughly speaking, the nonsense-logic systems are systems of three-valued logic with the following logical values: truth, falseness and nonsense, where the word "nonsense" may be understood in different ways. In that logic we make distinction between propositions and sentences: a proposition may be true, false or nonsensical whereas a sentence may be either true or false. The main idea of this paper is the following: let us introduce a new kind of many-valued logic ("stretched" over the three basic logical values: truth, falseness, nonsense) and use it for construction of a new generation of fuzzy sets. It is however necessary to answer the questions:

(a) how to interprete the words "nonsense" or "nonsensical"?
(b) which propositions may be called "nonsensical"?
(c) how to define logical operations in that new many-valued logic?
(d) how to define main operations for that new generation of fuzzy sets?

On the Seminar on Fuzzy&Interval Mathematics (see Acknowledgements) a suitable heated discussion is carried on. Its results will be published in a near future.

2. On nonsensical propositions

Now we shall attempt to answer the questions 1(a), 1(b). In the above-mentioned semimonograph /11/ some reasonable interpretations of the notion "nonsensical proposition" (NP) are reviewed. It appears that each of them induces a new nonsense-logic system.

Let := stand for "means". The following interpretations are presented in /11/:

(i) NP:= a proposition whose construction is incompatible with the Russell's theory of logical types,

e.g. "the set A is an element of the set A" - the so-called Russell's antinomy. The suitable nonsense-logic systems were defined by Bochvar /2/, Hallden /6/ and Finn /4/. The purpose of those systems is to omit logical antinomies - they are, simply, added to the logic.

(ii) NP:= a metaphysical or normative proposition.

The suitable nonsense-logic system was created by Hallden/6/ (cf. also Segerberg /12/).

(iii) NP:= an imperative proposition.

Some detailed informations about the nonsense-logic system are placed in Åqvist/1/ (see also /7/).

(iv) NP:= a propositional formula which lost its, say, numerical sense for some values of variables.

An outline of suitable nonsense-logic system is given in /11/,/13/. That case of NP is of great importance because the well-known equalities

$\{x:\ q(x) \vee s(x)\} = \{x:\ q(x)\} \cup \{x:\ s(x)\}$,

$\{x:\ \sim p(x)\} = \{x:\ p(x)\}^c$, where A^c denotes the complement of A, do not hold if q(x) (or s(x)), p(x) lose the numerical sense for some x. For instance, if $A=\{x \in R:\ 1/x>0\}$, $A^{\sim}=\{x \in R:\ \sim(1/x>0)\}$, then $A^{\sim}=(-\infty,0)$ and $A^c=(-\infty,0]$, $\{x \in R:\ 1/x>0 \vee x\leq 0\}=R-\{0\}$ and $\{x \in R:\ 1/x>0\} \cup \{x \in R:\ x \leq 0\}=R$.

We like to supplement the list by

(v) NP:= a ludicrous proposition,

e.g. "1 dollar a month is a good salary". This way we can get an emotional theory of falseness.

(vi) NP:= a proposition whose construction is incompatible with syntactical or semantical rules of a given formal or natural language,

for instance:

(a) ciphers (e.g. tw h n1zzw ct)
(b) some texts of modern poetry(esp. without footnotes)
(c) propositions in which we attempt to attribute some properties to an object while those properties are not related to the object (in questionnaires we write in such the situation "not relevant"), e.g. "this snake is tall" (cf. the interpretation (iv))

(vii) NP:= a proposition which is neither true nor false but whose logical value lies "between" true and false (i.e. a lack of logical sense only in the scale "true or false").

This case collapses to the usual interpretation of the third logical value which may be denoted by 1/2.

Finally, let us notice that the classes of propositions generated by the definitions (i)-(vii) are not disjoin.

3. Fuzzy sets of a new generation

The third logical value is in the case 2(vii) an "internal" value in relation to 0(false) and 1(true). For others interpretations of nonsense (2(i)-2(vi)), the third value is an "external" one. Therefore we propose, being inspired by the paper /10/ written by P.J.MacVicar-Whelan, to denote it by -1. Thus we get a three-valued logic with the following logical values: 1(true), 0(false) and -1(nonsensical).

Now, we can "stretch" a new many-valued logic over 1,0,-1 and , finally, introduce a new generation of fuzzy sets such that any membership function is of the form $A: U \to [-1,1]$, where A(x) may be interpreted as the truth-value (nonsense--value, resp.) of the proposition "x is in A" provided that $A(x) \geq 0$ ($A(x) < 0$, resp.).

Remarks

(a) P.J.MacVicar-Whelan uses the scale $[-1,1]$ instead of $[0,1]$ but he do this rather for technical convenience. However, he wrote in /10/, p.508: "-1 represents 100 percent confidence that the label is false or <u>does not apply</u>".

(b) Using the scale $[-1,1]$ we must agree that nonsense is always the worse logical value than falseness and that each proposition being true (to any degree $d \geq 0$) is absolutely sensible (in a given meaning of the word). The author is of the opinion that such the assumption may be (from the viewpoint of applications) inconvenient in some situations.

Another possibility is to define the membership functions of the new generation as functions from U into $[0,1] \times [0,1]$. Then $A(x)=(a,b)$, where $a,b \in [0,1]$ and a (b, resp.) denotes the truth-value (nonsense-value, resp.). Using such the definition, the truth-value and the nonsense-value would be considered to be independent ones.

4. Some applications of the new generation of fuzzy sets

In this section we like to present ideas dealing with some applications of the new generation of fuzzy sets. To this end we shall again consider the cases 2(iv)-2(vi) which seem to be especially interesting from that viewpoint.

4.1. NP:= a propositional formula which lost its numerical sense for some values of variables.

First, let us consider the following crisp subsets of the real line: $A=\{x\in R: 1/x>0\}$, $B=\{x\in R: x\leq 0\}$. In the language of the classical set theory we at once get $A\cup B=R$, $A\cap B=\emptyset$,... . But it is a problem if we attempt to construct union, intersection,... in the language of fuzzy set theory. To be exact, the difficulty is that it is not clear how to define the membership value A(0), namely:

(a) Putting A(0)=0 we obtain A(x)=A(0)=0 for $x<0$, i.e. we make no distinction between, say, A(-1) and A(0) while A(-1)=0 implies $1/-1\leq 0$ and A(0)=0 does not imply the respective inequality because 1/0 is, numerically, nonsense. In general, it is then possible to give an erroneous interpretation of a membership value. Really, all is clear as long as we can observe the propositional formula related to a set (e.g. $1/x>0$ in the definition of A). Without that formula we are however "blind".

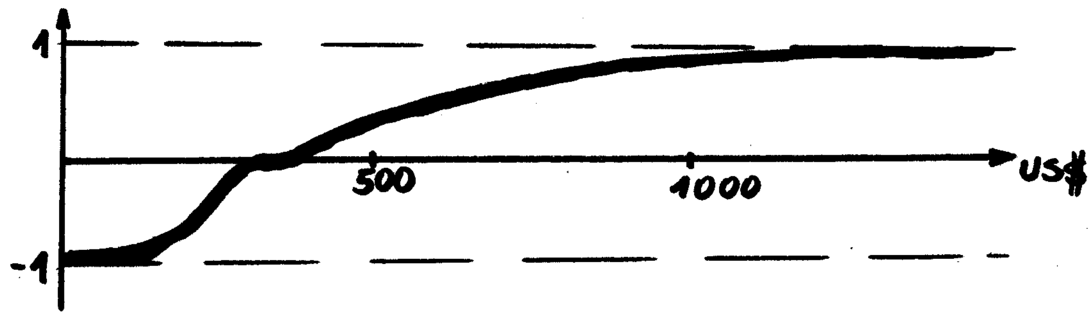
(b) If we define $A=\{x\in R-\{0\}: 1/x>0\}$, then A and B are subsets of different universal sets. Thus we can not construct $A\cup B$, $A\cap B$,... in the language of fuzzy sets theory.

We notice that an analogous problem will also occur for any fuzzy set which is defined by means of a formula losing its (e.g. numerical) sense for some values of variables (e.g. the fuzzy set composed of real numbers x such that $1/x\gg 0$). We can omit all the difficulties using fuzzy sets of the new generation. Then, for instance, if $A=\{x\in R: 1/x>0\}$, then A(x) may be defined (resp.) as 1,0,-1 for $x>0$, $x<0$, $x=0$ (resp.) and we get a deeper (more precise) information about A.

4.2. NP:= a ludicrous proposition.

Let us construct the fuzzy subset "good salary a month" of $(0,\infty)$. Using the classical version of fuzzy sets, the membership function may be subjectively defined in the following way:

We can agree that "200 US$ a month is a good salary" is false. "1 US$ a month is a good salary" is also false but, from the emotional point of view, we can say that this proposition is quite ludicrous (absurd). Using the new generation of fuzzy sets we can define the membership function as below.



Such the information about "good salary a month" seems to be deeper and more consistent with our emotions.

4.3. NP:= a proposition in which we attempt to attribute some properties to an object while those properties are not related to the object.

We limit ourself to the case 2(vi)(c) because it is the most convenient one.

Let us notice that usual fuzzy subsets (i.e. functions $U \rightarrow [0,1]$) are always subsets of a homogeneous universal set U, i.e. all the elements of U are of the same type (e.g. U composed of men or real numbers). Let us take into account a more general case when U is composed of elements of various types (a heterogeneous universal set). If we like to construct a fuzzy subset consisting of elements (from U) which fulfil some condition (have some property), it is possible that this property is not related to some elements from U.

Example.

Let U={Mr Smith, Mr Brown, Mr Taylor, a dog called Ace, sheet of paper} and A denote the fuzzy subset (of U) composed of elements which are higherly educated. Suppose that Mr Smith is a graduate of the Oxford University, Mr Brown terminated his study at an university, Mr Taylor got only the certificate of an elementary school and that Ace is a trained dog. If A: U→[0,1], we can at once write A(Mr Smith)=1,A(Mr Brown)=0.5 and A(Mr Taylor)=0. Willy-nilly we must put A(Ace)=0 and A(sheet of paper)=0. This way, however, we make no distinction between Mr Taylor, Ace and sheet of paper. On the other hand, the words "education", "educated" are not related to paper as well as to dogs (these words may be eventually used instead of "training", "trained"). Using the new generation we can omit that inconvenience, namely we put A(Ace)=-0.5 (because Ace is trained), A(sheet of paper)=-1.

The author realizes that the above-given example is probably the extreme one. On the other hand, an analogous problem will always occur in case of any heterogeneous universal set. Therefore fuzzy sets of the new generation may be an useful tool (for instance, if we like to construct a knowledge representation system and systematize our knowledge about objects of various types).

The problem of union, intersection,... is still open for the new fuzzy sets considered in 4.1-4.3. Some suggestions, as we previously mentioned, will be published in a near future.

5. Concluding remarks

We must emphasize the fact that the presented new generation of fuzzy sets is indeed an alternative concept for usual fuzzy sets but it is not the alternative of the type "either one or the other". The author is of the opinion that in some cases of applications the new fuzzy sets may be, simply, a more subtle tool than the classical fuzzy sets. However, the problem of heterogeneous universal sets seems to be solvable only by means of that new generation.

Fuzzy languages were introduced as a "bridge" between formal and natural languages. But apart from precise and fuzzy propositions, in practice of any natural language we sometimes meet propositions of a third kind, namely propositions which are (at least subjectively) nonsensical in a given meaning of the word. Therefore fuzzy languages based on the new generation of fuzzy sets would be a better link between formal and natural languages.

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