OPTIMUM FUZZY IMPLICATION AND DIRECT METHOD OF APPROXIMATE REASONING

Chen Yong-yi
Section of Math., Beijing Meteorological College, Beijing, China.

Wang Peizhuang
Department of Math., Beijing Normal University, Beijing, China.

Introduction

In [1] L.A. Zadeh suggested a method of approximate reasoning called "compositional rule of inference" (write CRI), which may be represented as

$$\frac{\begin{array}{l} A \longrightarrow B \\ A^* \end{array}}{B^* = A^* \circ (A \longrightarrow B)} \qquad \text{(modus ponens, write MP)}$$

and

$$\frac{\begin{array}{l} A \longrightarrow B \\ B^* \end{array}}{A^* = (A \longrightarrow B) \circ B^*} \qquad \text{(modus tollens, write MT)}$$

where $A, A^* \in \mathcal{F}(U)$, $B, B^* \in \mathcal{F}(V)$, $(A \longrightarrow B) \in \mathcal{F}(U \times V)$ and "o" is the "sup—$\wedge$" composition operation.

The fuzzy implication proposition $A \longrightarrow B$ defined by L.A.Zadeh as:

$$A \longrightarrow B \triangleq R \in \mathcal{F}(U \times V)$$

where $R(u,v) = [1 - A(u)] \vee [A(u) \wedge B(v)]$ or $R(u,v) = 1 \wedge [1 - A(u) + B(v)]$

Morever E.H.Mamdani[2], W.Bandler &L.Kohout[3], R.Willmott[4] and M.Mizumoto[5] in succession suggested some difference definition of fuzzy implication relations, they are R_m, R_a, R_c, R_s, R_g, R_b, R_{sg}, R_{gg}, R_{gs}, R_{ss}, $R_\triangle$, $R_\blacktriangle$, R_*, $R_\#$, $R_\square$ (see [5]), etc.. When using the method CRI to approximate reasoning, we always need a fuzzy relation $R \in \mathcal{F}(U \times V)$, which depend on A and B. Such the method CRI is not convenient to practical application.

In classical two valued logic, the logicol reasoning is "abstractness", i.e. the conclusion of reasoning is dependet only on the truth values (0 or 1) of propositions A and B.

An "abstractness" approximate reasoning method with fuzzy truth values and the Łukasiewicz's definition of implication is given by J.F.Baldwin in [7] (we write TVR), which may be represented as Fig.1:

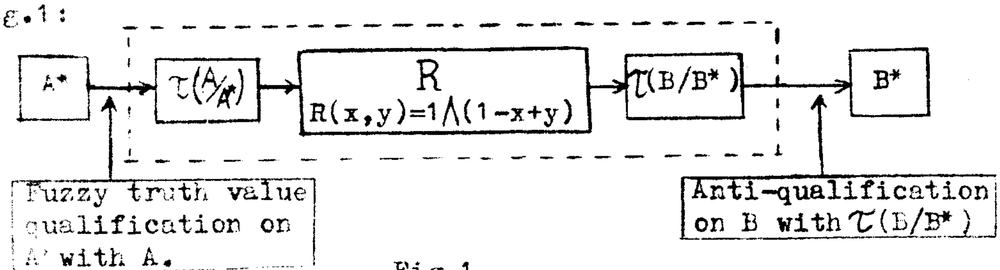


Fig.1

In above, $\tau(A/A^*)$, $\tau(B/B^*) \in \mathcal{F}(I)$, $I = [0,1]$

$$\tau(A/A^*)(x) = \sup_{A(u)=x} A^*(u) \qquad (1)$$

$$\tau(B/B^*)(y) = \tau(A/A^*)(x) \circ R(x,y) = \sup_{x\in I}\{\tau(A/A^*)(x) \wedge [1 \wedge (1-x+y)]\} \quad (2)$$

and
$$B^*(v) = \tau(B/B^*)[B(v)] \quad (3)$$

We point out, that the part of Fig.1 in dush line has more "abstractness", and it is just the advantage of method TVR over the CRI.

Note that, the method TVR has generality. Similar to CRI, we can give some difference definitions of $R(x,y)$ as R_m, R_α ... etc..

1. Equivalence of two methods

In this section we proved the approximate reasoning method TVR is equivalent to the method CRI. The mean of equivalence is such, that from same premises (major and minor) we can draw one and same conclusions.

Theorem 1. Assume $R_1(u,v) = R[A(u),B(v)] \in \mathcal{F}(U\times V)$, $R_2(x,y) \in \mathcal{F}(I\times I)$ are the fuzzy implication relations of method CRI and TVR, respectively.

If $R_2(x,y)=R(x,y)$, and the range of $A(u)$ is I, the two methods of approximate reasoning are equivalence.

Proof: We only prove to MP, the MT similarly.

Let $A, A^* \in \mathcal{F}(U)$, $B \in \mathcal{F}(V)$ and $A \longrightarrow B$.

From method CRI, we have

$$\forall v\in V \quad B_1^*(v) = A^*(u) \circ R_1(u,v) = \sup_{u\in U}[A^*(u) \wedge R_1(u,v)] = \sup_{u\in U}[A^*(u) \wedge R(A(u),B(v))] \quad (4)$$

From method TVR, by (1) we have

$$\tau(A/A^*)(x) = \sup_{A(u)=x} A^*(u), \quad (x\in I)$$

Because the range of $A(u)$ is I, so $\tau(A/A^*) \in \mathcal{F}(I)$ is a fuzzy truth value. Thus by (2) we have

$$\tau(B/B^*)(y) = \tau(A/A^*)(x) \circ R_2(x,y) = \tau(A/A^*)(x) \circ R(x,y) = \sup_{x\in I}\{\sup_{A(u)=x}[A^*(u) \wedge R(A(u),y)]\} = \sup_{u\in U}[A^*(u) \wedge R(A(u),y) \qquad (y\in I)$$

and by (3) we have

$$\forall v\in V \quad B_2^*(v) = \tau(B/B^*)(B(v)) = \sup_{u\in U}[A^*(u) \wedge R(A(u),B(v))] \quad (5)$$

So by (4) (5) we obtain

$$\forall v\in V \quad B_1^*(v) = B_2^*(v)$$ Q.E.D.

Lastly, let us notice that if $R(x,y)=1\wedge(1-x+y)=R_2(x,y)$, then we obtain the equivalence of method L. A. Zadeh's and L. F. Baldwin's.

2. Optimum fuzzy implication

In this section we make an approach to differece fuzzy implication, and find out the optimum.

Firstly, we point out, that the fuzzy truth values "TRUE" and "FALSE" are not the "TRUE" and "FALSE" of classical two valued logic. The "TRUE" of two valued logic is "ABSOLUTELY TRUE", and defined as

$$\text{"ABSOLUTELY TRUE"} \triangleq T(x)=\begin{cases}0 & 0\leq x<1\\ 1 & x=1\end{cases}$$

similarly, the "FALSE" is "ABSOLUTELY FALSE" and as

$$\text{"ABSOLUTELY FALSE"} \triangleq F(x)=\begin{cases}0 & 0< x\leq 1\\ 1 & x=0\end{cases}$$

If we given a partial order relation in $\mathcal{F}(I)$ as following:

$$\tau_1,\tau_2\in \mathcal{F}(I),\ \tau_1\leq\tau_2 \Longleftrightarrow \widetilde{\max}(\tau_1,\tau_2)=\tau_2,\ \text{(the } \widetilde{\max} \text{ see [8] P.52)}$$

then the T(x) and F(x) just are maximum element and minimum element of fuzzy truth valued set.

In approximate reasoning following properties of two valued logic must keep up:

Property 1. If $A\rightarrow B$ and A is the "ABSOLUTELY TRUE", then B is the "ABSOLUTELY TRUE";

Property 2. If $A\rightarrow B$ and B is the "ABSOLUTELY FALSE", then A is the "ABSOLUTELY FALSE";

Assume $R\in\mathcal{F}(I\times I)$ is a fuzzy implication relation of approximate reasoning, we have:

Theorem 2. The property 1 holds if and only if $R(1,y)=T(y)$;
The property 2 holds if and only if $R(x,0)=F(x)$;

Proof: Property 1 holds $\Longleftrightarrow T(x)\circ R(x,y)=T(y)$

$$\Longleftrightarrow \sup_{x\in I}\left[T(x)\wedge R(x,y)\right]=T(y)$$

$$\Longleftrightarrow R(1,y)=T(y)$$

Property 2 holds $\Longleftrightarrow R(x,y)\circ F(y)=F(x)$

$$\Longleftrightarrow \sup_{x\in I}\left[R(x,y)\wedge F(y)\right]=F(x)$$

$$\Longleftrightarrow R(x,0)=F(x) \qquad \text{Q.E.D.}$$

From this theorem the boundary restriction of a fuzzy implication relation is given, we called "boundary conditions" (Fig.2). Note that to R_b, R_c, R_{sg}, R_{gg}, R_{gs}, R_{ss} (in [5]) some boundary conditions are destroyed.

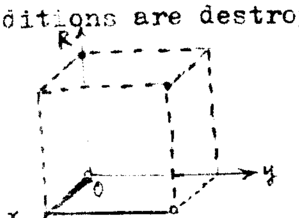


Fig.2

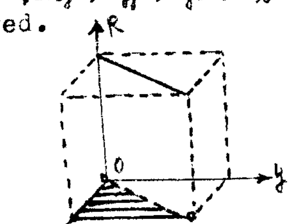


Fig.3

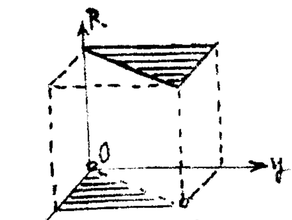


Fig.4

In approximate reasoning we hope hold the "co-ordination of mood", i.e. if $A \rightarrow B$, then "very $A \rightarrow$ very B", "fairly $A \rightarrow$ fairly B" etc.. Usualy the operations of mood is H_λ as

$$H_\lambda[A(x)] = [A(x)]^\lambda \quad (\lambda=1,2,\dots,n;\tfrac{1}{2},\tfrac{1}{4},\dots\tfrac{1}{n}.)$$

thus above conditions of mood may be represented as:

<u>Property 3.</u> If $A \rightarrow B$, then $H_\lambda(A) \rightarrow H_\lambda(B)$.

We call above property to the condition of mood, and have:

<u>Theorem 3.</u> For arbitrary monotono increasing fuzzy truth value $f \in \mathcal{F}(I)$, $\forall y \in I$, $f(x) o R(x,y) = f(y) \Longleftrightarrow R(x,y) = \begin{cases} 0 & x > y \\ 1 & x = y \end{cases}$

(Note that, where we do not restricte the value of $R(x,y)$ at $x < y$, see Fig.3.)

<u>Proof:</u> "$\Longleftarrow$" Assume $R(x,y) = \begin{cases} 0 & x > y \\ 1 & x = y \end{cases}$ and $f(x)$ is a monotono increasing fuzzy truth value, then

$$\forall y \in I, \quad f(x) o R(x,y) = \sup_{x \in I} [f(x) \wedge R(x,y)]$$
$$= \sup_{\substack{x \in I \\ x < y}} [f(x) \wedge R(x,y)] \vee f(y)$$

Since
$$\sup_{\substack{x \in I \\ x < y}} [f(x) \wedge R(x,y)] \leqslant \sup_{\substack{x \in I \\ x < y}} f(x) \leqslant f(y)$$

so
$$f(x) o R(x,y) = f(y)$$

"$\Longrightarrow$" Reductio ad absurdum. Assume for any monotono increasing $f \in \mathcal{F}(I)$, have $f(x) o R(x,y) = f(y)$, but $R(x,y)$ do not satisfy the conditions, i,e. or 1) $\exists\, x_0 > y_0$ such $R(x_0, y_0) = k_1 > 0$;
or 2) $\exists\, x_0 = y_0 = K$ such $R(K,K) = k_2 < 1$ $(k_2 \neq 0)$
under all circumstances we can found a $f(x)$ such, that $f(x) o R(x,y) \neq f(y)$, so contradictory.

For 1), let
$$f_1(x) = \begin{cases} 0 & 0 < x \leqslant y_0 \\ k_1 & y_0 < x \leqslant 1 \end{cases}$$

then
$$f_1(x) o R(x, y_0) = \sup_{x \in I} [f_1(x) \wedge R(x, y_0)]$$
$$\geqslant f_1(x_0) \wedge R(x_0, y_0)$$
$$\geqslant k_1 \wedge k_1 = k_1 > f_1(y_0) = 0, \text{ contradictory.}$$

For 2), let
$$f_2(x) = \begin{cases} 0 & 0 \leqslant x < K \\ 2k_2 & K \leqslant x \leqslant 1 \end{cases}$$

then
$$f_2(x) o R(x,K) = \sup_{x \in I} [f_2(x) \wedge R(x,K)]$$
$$= \Big[\sup_{x<K}(\cdots)\Big] \vee \Big[\sup_{x>K}(\cdots)\Big] \vee \Big[f(K) \wedge R(K,K)\Big]$$

since if $x < K$ have $f_2(x)=0$; if $x > K$ have $R(x,K)=0$
so that $\quad f_2(x) o R(x,K) = f(K) \wedge R(K,K) = 2k_2 \wedge k_2 = k_2$
but $f_2(K) = 2k_2$, thus $f_2(x) o R(x,K) < f_2(K)$, contradictory, Q.E.D.

It is generally believed that the fuzzy implication relation $R(x,y)$ is monotono increasing function of y and monotono decreasing fonction of x, so have

<u>Property 4.</u> A fuzzy implication relation $R(x,y)$ is nondecreasing of y and nonincreasing of x function.

The optimum fuzzy implication is a fuzzy relation, which satisfy above **four** properties. Thus we obtain a basic theorem:

<u>Theorem 4.</u> The optimum fuzzy implication is and only is the

$$R_S(x,y) = \begin{cases} 0 & x > y \\ 1 & x \leqslant y \end{cases} \quad \text{(as Fig. 4)}$$

Proof: Obviously, from theorem 2,3 and property 4 .
It is of interest to note that, the optimum fuzzy implication just is a nonfuzzy relation, which is the "Standard sequence" implication of many valued logic.

4. Direct method of approximate reasoning

In this section a new method of approximate reasoning with R_s is given.

Lemma 1. Let $R_s=\begin{cases}0 & x>y\\ 1 & x\leq y\end{cases}$, $f(x)$, $g(x)$ are monotono increasing and decreasing fuzzy truth values, respectively. we have

1) $\forall y\in I,\quad f(x)oR_s(x,y)=f(y)$
2) $\forall y\in I,\quad g(x)oR_s(x,y)=g(0)$

Proof: The 1) is a corollary of theorem 3. We only prove 2).
At arbitraly $y\in I$,

$$g(x)oR_s(x,y)=\sup_{x\in I}[g(x)\wedge R_s(x,y)]$$
$$=\sup_{x\leq y}[g(x)\wedge 1]$$
$$=\sup_{x\leq y}\ g(x)$$

from the monotono decreasing, we have

$$g(x)oR_s(x,y)=g(0) \qquad \text{Q.E.D.}$$

Let $\tau(x)$ is a arbitraly convex fuzzy truth value, and

$$\tau(x_0)=\max_{x\in I}[\tau(x)]$$

then it may be represented as

$$\forall x\in I,\quad \tau(x)=\tau_L(x)\wedge\tau_R(x) \qquad (6)$$

where $\tau_L(x)=\begin{cases}\tau(x) & 0\leq x\leq x_0\\ \tau(x_0) & x_0\leq x\leq 1\end{cases}$ is increasing and

$\tau_R(x)=\begin{cases}\tau(x_0) & 0\leq x\leq x_0\\ \tau(x) & x_0\leq x\leq 1\end{cases}$ is decreasing (see Fig.5).

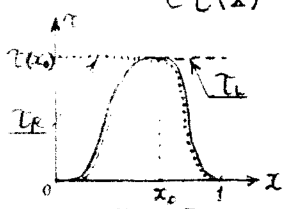


Fig.5

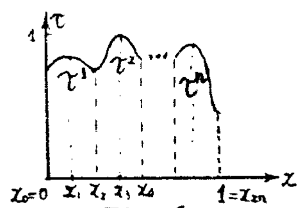


Fig.6

Lemma 2. Assume $\tau(x)$ is a convex fuzzy truth value as (6), then $\forall y\in I,\quad \tau(x)oR_s(x,y)=\tau_L(y)$

Proof: If $y\leq x_0$, then $\tau(x)oR_s(x,y)=\sup_{x\in I}[\tau_L(x)\wedge\tau_R(x)\wedge R_s(x,y)$

$$=\sup_{x\leq y}\tau_L(x)$$
$$=\tau_L(y)$$

If $y\geq x_0$, then $\tau(x)oR_s(x,y)=\sup_{x\in I}[\tau_L(x)\wedge\tau_R(x)\wedge R_s(x,y)]$

$$=\sup_{x\leq y}\tau_R(x)$$
$$=\tau_R(x_0)=\tau_L(x_0)$$

$$\therefore\quad \tau(x)oR_s(x,y)=\tau_L(y)=\begin{cases}\tau_L(y) & 0\leq y\leq x_0\\ \tau_L(x_0) & x_0\leq y\leq 1\end{cases} \qquad \text{Q.E.D.}$$

The lemma 2 illutrate, that in approximate reasoning with R_S the used part of a convex fuzzy truth value $\tau(x)$ only is its increasing part $\tau_L(x)$. This is importance.

Let $\tau(x)$ is a no-convex fuzzy truth value as Fig.6,

we have $$\forall x \in I \quad \tau(x)=\bigvee_{i=1}^{n}\tau^i(x) \tag{7}$$

where $$\tau^i(x)=\begin{cases}\tau(x) & x_{2i-2}\leq x \leq x_{2i} \\ 0 & \text{otherwise}\end{cases}$$
$$=\tau_L^i(x)\wedge\tau_R^i(x) \qquad (i=1,2,\ldots,n.)$$

<u>Theorem 5.</u> Assume $\tau(x)$ is a no-convex fuzzy truth value as (7), then $\forall y \in I \quad \tau(x) o R_S(x,y)=\bigvee_{i=1}^{n}\tau_L^i(y)$

<u>Proof:</u> From the distributive law of operation "o" to "$\cup$" and lemma 2 it is proved.

<u>Corollary.</u> In theorem 5, if
$$\operatorname*{Max}_{x\in I}[\tau^1(x)]=\operatorname*{Max}_{x\in I}[\tau^2(x)]=\cdots\cdots=\operatorname*{Max}_{x\in I}[\tau^n(x)]$$
then $\quad \forall y \in I \qquad \tau(x) o R_S(x,y)=\tau_L^1(y)$

<u>Proof:</u> Obviously.

From theorem 5 and corollary we obtain a new method of approximate reasoning——direct method, there needs no the composition operation of fuzzy relation.

The general steps of direct mehtod following:

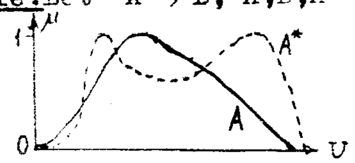
Give $A, A^*\in\mathcal{F}(U)$, $B\in\mathcal{F}(V)$ and $A\rightarrow B$, find the $B^*\in\mathcal{F}(V)$.

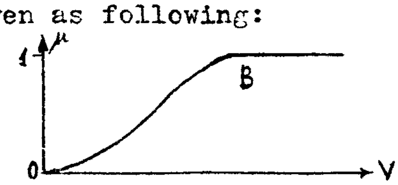
1) Qualification: From A and A^* calculate the $\tau(A/A^*)$ as
$$\tau(A/A^*)(x)=\operatorname*{SUP}_{A(u)=x} A^*(u)$$

2) Truth valued reasoning: Express $\tau(A/A^*)$ as (7), and we by theorem 5 obtain:
$$\forall y \in I \quad \tau(B/B^*)(y)=\bigvee_{i=1}^{n}\tau_L^i(y)$$

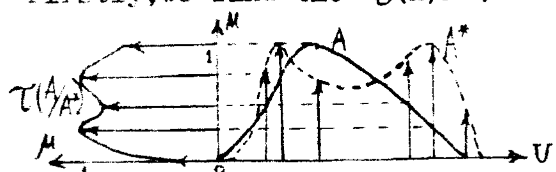
3) Anti-qualification: $B^*(v)=\tau(B/B^*)[B(v)]$

<u>Example:</u> Let $A\rightarrow B$, A, B, A^* is given as following:

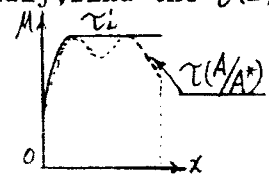




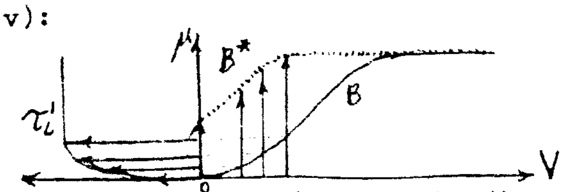
Firstly, we find the $\tau(A/A^*)$:



Secondly, find the $\tau(B/B^*)=\tau_L^1$:



Lastly, obtain the $B^*(v)$:



In above we have discussed on the MP, for MT the theorem 5 need only turn τ_L^i into τ_R^i.

5. About SUP-T reasoning

In this section we discuss the approximate reasoning under "SUP-T" composition operation. The T is a triangular norm, which as a intersection operator of fuzzy subsets.([9] [10])

A matter for rejoicing is that, the T-norms satisfies following conditions:

$$T(x,0)= 0 = x \wedge 0$$
$$T(x,1)= x = x \wedge 1$$

and the $R_5(x,y)$ is a nonfuzzy relation, i.e. for arbitrary $x,y \in I$, have $R_5(x,y) \in \{0,1\}$, so that

$$\sup_{x \in I} [A(x) T R_5(x,y)] = \sup_{x \in I} [A(x) \wedge R_5(x,y)]$$

Thus we obtain a unforseen result: all conclutions of "SUP-$\wedge$" reasoning with R_5 are suitable to "SUP-T" approximate reasoning. Certainly, the direct method is suitable too.

Conclusion

We proved the equivalence of both method CRI and TVR, proved the "optimum fuzzy implication" is one and only, it is the implication "Standard sequence" of many valued logic R_5.

The direct method of approximate reasoning is given, this method is simple and very good for multiple and compound implication. Application of the method can be made to such areas as fuzzy control, medical diagnosis, artificial intelligence, decision theoty, etc..

References

[1] L.A.Zadeh, Outline of a new approach to the analysis of complex systems and decision processes, IEEE Trans. Systems, Man and Cybernet, 1,(1973),28-44.

[2] E.H.Mamdani, Application of fuzzy logic to approximate reasoning using linguistic systems, IEEE Tras. Comput.,C-26 (1977), 1182-1191.

[3] W.Bandler & L.Kohout, fuzzy power sets and fuzzy implication operators, Fuzzy Sets and Systems, Vol.4, Num.1(1980), 13-30.

[4] R.Willmott, Two fuzzier implication operators in the theory of fuzzy power sets, Fuzzy Sets & Sys., Vol.4, No.1(1980)31-37.

[5] M.Mizumoto, H.J.Zimmerman, Comparison of fuzzy reasoning methods, Fuzzy Sets & Sys., Vol.8, No.3,(1982) 253-283.

[6] N.Rescher, Many Valued Logic (McGraw-Hill, New York, 1969).

[7] J.F.Baldwin, A new approach to approximate reasoning using a fuzzy logic, Fuzzy Sets & Sys., Vol.2,(1979),309-325.

[8] D.Dubois &H.Prade, Fuzzy Sets and Systems, 1980.

[9] B.Schweizer, A.Sklar, Associative functions and abstract semi-groups, Publ. Math. Debrecen, 10,(1963), 69-81.

[10] Chen Yong-yi, An approach to fuzzy operators, BUSEFAL, (1981,1982),9.