DUALITY IN FUZZY LINEAR PROGRAMMING

J.L. Verdegay
Departamento de Estadistica Matematica
Facultad de Ciencias
Universidad de Granada
GRANADA (SPAIN)

INTRODUCTION.- In this work, we consider, exclusively, two types of possible Fuzzy Linear Programming (FLP) problems:

$$\begin{array}{l} \text{Max: } z = cx \\ \text{s.t:} \\ \qquad Ax \underset{\sim}{\leq} b \\ \qquad x \underset{\sim}{\geq} 0 \end{array} \qquad (P1)$$

(with only fuzziness on the constraints) and,

$$\begin{array}{l} \text{Max: } z = cx \\ \tilde{\text{s.t}}: \\ \qquad Ax \underset{\sim}{\leq} b \\ \qquad x \underset{\sim}{\geq} 0 \end{array} \qquad (P2)$$

(with a fuzzy objective in the sense of [10]).

Based on the fact that in Fuzzy Mathematical Programming in general, objectives and constraints plays the same role, - [1]: The dual of an FLP problem with fuzzy objective, must be an FLP problem with a fuzzy constraint set, and reciprocally. Essentially, this previous view states that the dual concept of fuzzy constraint is the fuzzy objective, and reciprocally. Nevertheless, the correspondence between FLP problems stated by the above idea of duality is too broad. The following result makes it more definite and states us the form which has the dual of an FLP problem in a more accurate way.

RESULT 1.- Given an FLP problem, P1 or P2, there is always - such one that both are dual and, besides, they have the same fuzzy solution.

Proof: Suppose start from an FLP problem such as,

$$\begin{aligned} \text{Max: } & cx \\ \text{s.t: } & \\ & Ax \lesssim b \\ & x \geq 0 \end{aligned} \qquad (1)$$

being $c \in R^n$, $b \in R^m$ and $A(m \times n)$ known matrices of real values.

If we state the fuzziness of the constraint set by means of an m-vector of membership functions $\mu = (\mu_1, \ldots, \mu_m)$ given- by,

$$\forall v \in R, \quad \mu_j(v) = \begin{cases} [(b_j + d_j) - v]/d_j & : b_j + d_j \geq v \geq b_j, \ j=1,\ldots,m \\ 1 & : \quad v < b_j \\ 0 & : b_j + d_j < v \end{cases} \qquad (2)$$

where, as usual, the values $d_j \in R$ $(j=1,\ldots,m)$ express the violations which the decision-maker allows in the accomplishment of the linear constraints of (1); we know that the fuzzy solution of (1) is found by obtaining the optimal solution of the linear parametric problem,

$$\begin{aligned} \text{Max: } & cx \\ \text{s.t: } & \\ & \mu(Ax, b) \geq \alpha \\ & x \geq 0, \ \alpha \in [0,1] \end{aligned}$$

But, according to (2),

$$\mu(Ax, b) \geq \alpha \leftrightarrow Ax \leq b + d(1-\alpha)$$

the whole expressed in a matrix form.

Therefore, we have

$$\begin{aligned} \text{Max: } & cx \\ \text{s.t: } & \\ & Ax \leq b + d(1-\alpha) \\ & x \geq 0, \ \alpha \in [0,1] \end{aligned} \qquad (3)$$

As this problem is a classical parametric linear programming problem, its dual is given by,

$$\begin{aligned} &\text{Min: } [b + d(1-\alpha)]\cdot u \\ &\text{s.t:} \\ &\qquad uA' \geq c \\ &\qquad u \geq 0,\ \alpha\in[0,1] \end{aligned} \tag{4}$$

If,

$$Y = \{u\in R^m / \ uA' \geq c,\ u \geq 0\}$$

we have,

$$\begin{aligned} &\text{Min: } au \\ &\text{s.t:} \\ &\qquad a = b + d(1-\alpha) \\ &\qquad u\in Y,\ \alpha\in[0,1] \end{aligned} \tag{5}$$

taking $\beta=1-\alpha$, this problem is equivalent to

$$\begin{aligned} &\text{Min: } au \\ &\text{s.t:} \\ &\qquad a \leq b + d(1-\alpha) \\ &\qquad u\in Y,\ \alpha\in[0,1] \end{aligned} \tag{6}$$

understanding the equivalence in the sense that every optimal solution of (5) is also an optimal solution of (6).

But as,

$$a_j \leq b_j+d_j\beta \leftrightarrow (b_j+d_j-a_j)/d_j \geq 1-\beta,\ j=1,\ldots,m$$

(6) may be rewritten as,

$$\begin{aligned} &\text{Min: } au \\ &\text{s.t:} \\ &\qquad \mu_j(a_j) \geq 1-\beta,\ j=1,\ldots,m \\ &\qquad u\in Y,\ \alpha\in[0,1] \end{aligned} \tag{7}$$

being $\mu_j(\cdot)$ given by (2). Thus, (7) is an FLP problem with fuzzy objective,

$$\underset{u\in Y}{\widetilde{\text{Min}}}\text{: } au \tag{8}$$

with membership functions in the objective coefficients given by (2). Furthermore, this FLP problem has the same fuzzy solution as (1), only by means of taking $\beta=1-\alpha$.

If we had initially started from a P2 FLP problem, developping it, in relation to the previous one, in a parallel -- way, we should have come to a P1 FLP problem with the same fuzzy solution as the one taken from the very start■

REMARK.- Taking into account this result, whenever the membership functions which take part in the statement of - the problem be like (2), we may define the dual of an FLP problem given by (1), as (7), or reciprocally, the dual of (8) - as problem (3).

However, it seems that a good definition of the dual problem may only be stated in similar cases to the previous one, i.e., with linear membership functions. Now we shall see that under little restrictive hipotheses, the above result is easily applied in general.

RESULT 2.- Given an P1 (P2) FLP problem with membership functions $\mu(\cdot) = [\mu_i(\cdot),\ldots,\mu_m(\cdot)]$ for the restrictions (costs), if these are continuous and strictly monotones,- either increasing or decreasing following the sense of the inequalities (depending on whether it maximizes or minimizes), there exists always another P2 (P1) FLP problem, dual of the- former, and of such a kind that both possess the same fuzzy - solution.

Proof: Let,

$$\mu_j : R \longrightarrow [0,1] \;,\; j = 1,\ldots,m$$

be continuous and strictly increasing functions for the FLP - problem,

$$\begin{aligned} &\text{Max: } cx \\ &\text{s.t:} \\ &\quad Ax \underset{\sim}{\leq} b \\ &\quad x \geq 0 \end{aligned}$$

we shall find their fuzzy solution from every α-cut of the fuzzy constraint set,

$$\mu(Ax,b) \geq \alpha, \; \alpha \in [0,1]$$

But according to the hipotheses,

$$\mu(Ax,b) \geq \alpha \leftrightarrow Ax \leq \psi(\alpha) = \mu^{-1}(\alpha)$$

and the proof it follows as in the Result 1■

CONCLUDING REMARKS.- The linearity of FLP problems which have been considered, is never lost by the -- special form which the membership functions may be have, even if these are non linear. This fact is only related to the parameter used, therefore the linearity of the problem is not affected.

The most important uselfuness derived from the dual relationship which has been shown in this work, lies in the ability of solving problems with fuzzy constraint set without letting this fuzziness affect said set. This can be obtained by mean of the dual, which make easy the problem.

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