CRITERIA FOR NON - INTERACTIVITY OF FUZZY LOGIC CONTROLLER RULES

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Abstract Usually a fuzzy logic controller is given by a set of relation equations or as the fuzzy union of a family of control rules. Considering this secind approach we discuss the problem if a controller constructed this way really works in accordance with these control rules. Unfortunately, this is not always the case. Therefore, our main concern here is to look for sufficient conditions which guarantee that a controller finally works in accordance with the rules he was constructed from.

1. Fuzzy Logic Controllers

A fuzzy logic controller - short: FLC - is some device that connects fuzzy subsets of an input space $\underline{U}$ with fuzzy subsets of an output space $\underline{V}$ (cf. /4/,/3/,/2/).

These fuzzy sets normally are considered as representing the meanings of linguistic values of a linguistic input resp. output variable. Hence, a FLC is a - finally: automatized - realization of a (simple) process of approximate reasoning.

Mathematically, such a FLC R is a fuzzy subset of the cartesian product $\underline{U} \times \underline{V}$, i.e. a fuzzy binary relation. And the fuzzy output B for a given fuzzy input A is given by the relation equation $B = A \circ R$, i.e. through

$$\mu_B(y) = \max_{x \in \underline{U}} \min(\mu_A(x), \mu_R(x,y)) \ . \qquad (1)$$

If we adopt the point of view that the generalized membership values $\mu_A(x), \mu_B(y)$ etc. are generalized truth values of a many valued membership relation ε, i.e. if we take

$$\mu_A(x) = [x \varepsilon A] \ , \qquad \mu_B(y) = [y \varepsilon B] \qquad (2)$$

with $[H]$ for the truth value of expression H of a generalized set theoretic language, we may write formula (1) as

$$[y \varepsilon B] = [\bigvee_x (x \varepsilon A \wedge (x,y) \varepsilon R)] \ , \qquad (3)$$

interpreting existential quantification $\bigvee$ and conjunction $\wedge$ of many valued logic as taking the supremum resp. minimum.

Yet, more is obvious now. As with crisp sets also for fuzzy sets any fuzzy binary relation can be considered as a fuzzy mapping (in the generalized sense not including uniqueness condition). Then, firstly, equations(3) reads as

$$B = R''A \ , \qquad (4)$$

i.e. B is the fuzzified full picture of A under R. And secondly, from (3) and many valued logic it is obvious that the interpretations of $\bigvee$ and $\wedge$, mentioned above, are not the only possible ones.

Hence, by (4) and especially by (3) we get a more general look at FLC's. Of course, as (3) describes the usual understanding of so-called fuzzy modus ponens of approximate reasoning, our present discussion gives also a coherent view of a lot of different realizations of that generalized inference procedure as discussed e.g. by MIZUMOTO/ZIMMERMANN /6/.

2. The Construction of FLC's

Surely, there is no uniform way to construct a FLC. But at present two strongly connected methods are mostly used. The starting point in each case is a (finite) set of control rules

$$A_i \Rightarrow B_i \ , \quad i \in I \tag{5}$$

connecting values A_i of an input variable with values B_i of an output variable. The FLC R to be constructed is intended to realize

$$B_i = R''A_i \quad \text{for all } i. \tag{6}$$

A first possibility to get R is to consider (6) as a system of equations with R as unknown fuzzy mapping/fuzzy relation, and to try to solve this system. In this direction yet only very few results are known: in most cases systems with one equation only have been considered.

As a second possibility to get R one chooses usually the fuzzy union of all the rules (5). To be able to do so, first one has to represent each one of the rules $A_i \Rightarrow B_i$ by a fuzzy subset of $\underline{U} \times \underline{V}$. Already here there exist a lot of different proposals; some of them are

$$A \Rightarrow B = A \times B \ , \tag{7}$$

$$A \Rightarrow B = (A \times B) \cup (\overline{A} \times \tilde{V}) \ , \tag{8}$$

$$A \Rightarrow B = (\overline{A} \times \tilde{V}) \cup (\tilde{U} \times B) \ . \tag{9}$$

For more such proposals cf. /5/,/6/. Here $\tilde{U},\tilde{V}$ are the fuzzy subsets of $\underline{U}$ resp. $\underline{V}$ with membership value 1 always; and the cartesian product $X \times Y$, the union $X \cup Y$, and the complement $\overline{X}$ of fuzzy sets X,Y are defined through

$$[(x,y) \in X \times Y] = [x \in X \wedge y \in Y] \ , \tag{10}$$

$$[z \in X \cup Y] = [z \in X \vee z \in Y] \ , \tag{11}$$

$$[z \in \overline{X}] = [\neg z \in X] = [z \notin X] ; \qquad (12)$$

$\wedge, \vee, \neg$ any conjunction, disjunction, negation of many valued logic.

For negation $\neg$ we will consider only the case

$$\neg t = 1 - t \qquad (13)$$

for every truth value t, identifying here and furthermore always the connectives of many valued logic with their corresponding truth value functions.

For conjunction and disjunction we discuss different cases, always assuming that one conjunction and one disjunction are connected through deMorgan formulas. Especially we will consider:

$$s \underline{\wedge} t = \min(s,t) , \qquad s \underline{\vee} t = \max(s,t) , \qquad (14)$$

$$s \,\&\, t = \max(0,s+t-1) , \qquad s \underline{\underline{\vee}} t = \min(1,s+t) , \qquad (15)$$

$$s \wedge t = \begin{cases} 0 & \text{if } s,t<1 \\ s \underline{\wedge} t & \text{otherwise} \end{cases} . \qquad (16)$$

Sometimes we have to consider more than one conjunction or disjunction at once, then we use indices; and if we use indices together with fuzzy union or fuzzy intersection of two fuzzy sets these set theoretic operations then are supposed to be characterized by the suitable many valued connective with same index à la (10),(11). $(\forall ..)$, $(\exists ..)$ denote quantification in our classical metalanguage.

Having "coded" the rules $A_i \Rightarrow B_i$ in such a way by fuzzy subsets of $\underline{U} \times \underline{V}$, the whole FLC R is defined by

$$R = \bigcup_i (A_i \Rightarrow B_i) , \qquad (17)$$

i.e. by

$$[(x,y) \in R] = [\bigvee_i ((x,y) \in (A_i \Rightarrow B_i))] . \qquad (18)$$

3. The Problem of Interactivity

Unfortunately, this second construction of a FLC R out of a family of generating rules $A_i \Rightarrow B_i$, $i \in I$ by (17),(18) sometimes /1/ is not in accordance with (6), i.e. there exist $j \in I$ such that $B_j \neq R"A_j = A_j \circ R$. Thus, if a FLC is part of a decision algorithm it may happen that the decision coming from B_j is different of that from $R"A_j$. This means, in the sense of control theory, that the decision-making rules (5) are interacting.

Definition 1: A generating family $A_i \Rightarrow B_i, i \in I$ of a FLC R is called interactive iff there exist an index $k \in I$ such that $B_k \neq R"A_k$; otherwise the family $A_i \Rightarrow B_i, i \in I$ is called non-interactive.

Because for fuzzy sets X,Y it holds true

$$X = Y \quad \text{iff} \quad X \subseteq Y \text{ and } Y \subseteq X \;, \tag{19}$$

the problem of non-interactivity of a generating family of a FLC splits.

Definition 2: Suppose R to be a FLC and $A_i \Rightarrow B_i, i \in I$ a generating family of R. We say that R has the superset property w.r.t. the given generating family iff

$$R"A_i \supseteq B_i \quad \text{for all } i \;; \tag{20}$$

and we say that R has the subset property w.r.t. this family iff

$$R"A_i \subseteq B_i \quad \text{for all } i \;. \tag{21}$$

By (3) and (18) we have for each $v \in \underline{V}$ immediately

$$[v \in R"B] = [\bigvee_x (x \in A \wedge \bigvee_k ((x,v) \in (A_k \Rightarrow B_k)))] \;. \tag{22}$$

Using $\alpha_1 \leq \bigvee_i \alpha_i$, hence $\sup_i \leq \bigvee_i$ for both of these quantifiers, and the monotonicity of $\wedge$ as natural properties of many valued existential quantification and conjunction, we get

$$(\forall v,t)(\exists u,k)([v \in B_t] \leq [u \in A_t \wedge (u,v) \in (A_k \Rightarrow B_k)]) \qquad (23)$$

as a sufficient condition for R to have the superset property; and we get

$$(\forall v,t,u,k)([v \in B_t] \geq [u \in A_t \wedge (u,v) \in (A_k \Rightarrow B_k)]) \qquad (24)$$

as a necessary condition for R to have the subset property. The domains of the variables which are quantified here are clear by context.

Assuming additionally $\bigvee_i \leq \sup_i$ for both quantifiers of (22), condition (24) will become sufficient for the subset property. Furthermore, for finite generating families $A_i \Rightarrow B_i, i \in I$ and a finite universe of discourse $\underline{U}$ for the input fuzzy sets, condition (23) now becomes a necessary one for the superset property of $R = \bigcup_i (A_i \Rightarrow B_i)$.

Therefore, in the following we will assume that each many valued existential quantifier is interpreted as supremum, that the generating families of the FLC's are finite, and that also the universes of discourse $\underline{U}, \underline{V}$ are finite. This last assumption is not obvious, but realized in most applications, and may be avoided in principle with more difficult proofs. Also we assume of all the many valued conjunctions and disjunctions their monotonicity, commutativity, and associativity, and $\alpha \wedge \beta \leq \alpha \leq \alpha \vee \beta$ for all these conjunctions $\wedge$ and disjunctions $\vee$.

Proposition 1: Suppose R to be given by the generating rules $A_i \Rightarrow B_i, i \in I$.

(i) A necessary condition for the superset property of R is

$$(\forall t)(\ \mathrm{hgt}(B_t) \leq \mathrm{hgt}(A_t)\ ) .$$

(ii) A sufficient condition for the superset property of R is

$$(\forall t)(\ B_t \subseteq (A_t \Rightarrow B_t)"A_t\ ) .$$

(iii) A necessary and sufficient condition for the subset property of R is

$$(\forall t,k)(\,(A_k \Rightarrow B_k)''A_t \subseteq B_t\,)\,.$$

The proofs are straightforward and shall not be considered in detail. The same will be the case with the proofs of the following propositions. The height hgt(X) of a fuzzy set X, used in (i), is the supremum of all membership values $\mu_X(z)$, i.e. $hgt(X) = [\bigvee_z (z \in X)]$.

<u>Definition</u> 3: Given two realizations R_i^1, R_i^2 of a controller rule $A_i \Rightarrow B_i$ as a fuzzy subset of $\underline{U} \times \underline{V}$, we call R_i^1 a <u>stronger version</u> of $A_i \Rightarrow B_i$ as R_i^2 or R_i^2 a <u>weaker version</u> of $A_i \Rightarrow B_i$ as R_i^1 iff $R_i^1 \subseteq R_i^2$.

<u>Proposition</u> 2: Suppose R_i^1, R_i^2 to be realizations of controller rule $A_i \Rightarrow B_i, i \in I$ of a generating family of FLC R^1, R^2.

(i) If R_i^2 is a weaker version as R_i^1 for each i and R^1 has the superset property, then also R^2 has the superset property.

(ii) If R_i^2 is a stronger version as R_i^1 for each i and R^1 has the subset property, then also R^2 has the subset property.

The different realizations of a controller rule $A \Rightarrow B$, presented in (7),(8),(9) now can be compared with respect to their strenght. Because of

$$A \times B \subseteq (A \times B) \cup (\overline{A} \times \widetilde{V}) \subseteq (\overline{A} \times \widetilde{V}) \cup (\widetilde{U} \times B) \qquad (25)$$

in case that all the fuzzy cartesian products are characterized by the same many valued conjunction and both fuzzy unions by the same disjunction, (7) gives the strongest and (9) the weakest version of $A \Rightarrow B$. Thus, in (25) the superset property is transmitted "from left to right" and the subset property "from right to left".

To get more concrete results now we will consider the cases given by formulas (7) to (9) in more detail. Always we are interested in necessary and in sufficient conditions for (23),(24) to hold true, assuming the family $A_i \Rightarrow B_i, i \in I$ as a generating one for the FLC R.

3.1. Case 1: $(A \Rightarrow B) = A \times B$

R has superset property, i.e. (23) holds true iff

$$(\forall v,t)(\exists u,k)([v \in B_t] \leq [u \in A_t \wedge_1 (u \in A_k \wedge_2 v \in B_k)]) \; . \qquad (26)$$

Proposition 3: Sufficient conditions for R to have the superset property are

(i) $(\forall t)(\ hgt(A_t) = 1\ )$.

(ii) for $\wedge_1 = \wedge_2 = \underline{\wedge}$: $(\forall t)(\ hgt(B_t) \leq hgt(A_t)\ )$.

Correspondingly, R has subset property, i.e. (24) holds true iff

$$(\forall v,u,t,k)([v \in B_t] \geq [u \in A_t \wedge_1 (u \in A_k \wedge_2 v \in B_k)]) \; . \qquad (27)$$

Proposition 4: Sufficient conditions for R to have the subset property are

(i) $(\forall t,k \neq t)(\ A_t \cap_1 A_k = \emptyset\ )$.

(ii) for $\wedge_1 = \wedge$: $(\forall t,k \neq t)(\ hgt(A_t \cap A_k) < 1\ )$.

(iii) for always $\alpha \wedge_1 (\beta \wedge_2 \gamma) \leq (\alpha \wedge_1 \beta) \wedge_2 \gamma$:

$(\forall t,k \neq t)(\ hgt(B_k) \wedge_2 hgt(A_t \cap_1 A_k) = 0\ )$.

Here, (iii) is relevant e.g. if $\wedge_1$ is distributive over $\wedge_2$.

3.2. Case 2: $(A \Rightarrow B) = (A \times B) \cup (\bar{A} \times \tilde{V})$

R has superset property iff

$$(\forall v,t)(\exists u,k)([v \in B_t] \leq [u \in A_t \wedge_1 ((u \in A_k \wedge_2 v \in B_k) \vee_1 u \in \bar{A}_k)]) \; . \qquad (28)$$

Proposition 5: Sufficient conditions for R to have superset property are

(i) each condition of Prop. 3 .

(ii) for $\wedge_2 = \&$, $\vee_1 = \underline{\underline{v}}$, $\wedge_1 \in \{\underline{\wedge}, \&, \wedge\!\!\!\wedge, \text{usual product}\}$:

$$(\forall t)(\exists k)(\ \mathrm{hgt}(B_t) \leq \mathrm{hgt}(A_t \cap_1 \overline{A}_k)\ )$$

and also

$$(\forall t)(\ \mathrm{hgt}(B_t) \leq \mathrm{hgt}(A_t)\ ) .$$

As before, the proof is straightforward. For (ii) one has to use: $\neg\alpha\ \underline{\underline{v}}\,(\alpha \,\&\, \beta) = \neg\alpha\ \underline{\mathbf{v}}\ \beta$. And now, R has the subset property iff

$$(\forall v,u,t,k)([v \in B_t] \geq [u \in A_t \wedge_1((u \in A_k \wedge_2 v \in B_k) \vee_1 u \in \overline{A}_k)]) . \qquad (29)$$

Proposition 6: Sufficient conditions for R to have subset property are

(i) for $\wedge_1 = \wedge\!\!\!\wedge$: $(\forall t,k \neq t)(\ \mathrm{hgt}(A_t \,\underline{\cup}\, (A_k \cup_1 \overline{A}_k)) < 1\ )$.

(ii) for $\wedge_2 = \&$, $\vee_2 = \underline{\underline{v}}$, $\wedge_1 \in \{\underline{\wedge}, \&, \wedge\!\!\!\wedge, \text{usual product}\}$:

$$(\forall t,k \neq t)(\ \mathrm{hgt}(\overline{B}_t) + \mathrm{hgt}(A_t \cap_1 \overline{A}_k) \leq 1\ )$$

and also

$$(\forall t,k \neq t)(\ \mathrm{hgt}(A_t) \wedge_1 \mathrm{hgt}(B_k) = 0\ ) .$$

3.3. Case 3: $(A \Rightarrow B) = (\overline{A} \times \widetilde{V}) \cup (\widetilde{U} \times B)$

R has superset property iff

$$(\forall v,t)(\exists u,k)([v \in B_t] \leq [u \in A_t \wedge_1 (u \in \overline{A}_k \vee_1 v \in B_k)]) . \qquad (30)$$

Proposition 7: Sufficient conditions for R to have superset property are

(i) each condition of Prop. 5 .

(ii) if $\wedge_1$ distributes over $\vee_1$:

$$(\forall t)(\ \mathrm{hgt}(B_t) \leq \mathrm{hgt}(B_t) \wedge_1 \mathrm{hgt}(A_t)\ ) .$$

Finally, R has subset property iff

$$(\forall v,u,t,k)([v \varepsilon B_t] \geq [u \varepsilon A_t \wedge_1 (u \varepsilon \bar{A}_k \vee_1 v \varepsilon B_k)]) . \qquad (31)$$

Proposition 8: A necessary condition for R to have subset property is

$$(\forall t)(\mathrm{hgt}(\bar{B}_t) + \mathrm{hgt}(A_t \cap_1 \bar{A}_t) \leq 1) .$$

4. Concluding Remarks

In discussing different realizations R of that procedure of approximate reasoning that generalizes usual modus ponens, MIZUMOTO /5/ was considering unary operators $\underline{C}$ - represented e.g. by hedges like: very, more or less,... - and asking to have realized

$$\underline{C}(R''A) = R''(\underline{C}A)$$

for all fuzzy sets A.

As another point of view here we are interested to have realized

$$B_i = R''A_i$$

for a given family $A_i \Longrightarrow B_i, i \in I$ of pairs of fuzzy sets. With R as a FLC this is the problem of non-interactivity. For some different ways to construct R from the family of all rules $A_i \Longrightarrow B_i$ we got sufficient conditions for non-interactivity. These conditions in almost all cases are strong ones, indicating also some preferences of our case 1: $(A \Longrightarrow B) = A \times B$ over the other ones.

Generally, now it seems interesting to weaken the problem and to ask for estimations of (some suitable kind of) difference of B_i and $R''A_i$ for different R's. But this is another problem.

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