RESOLUTION OF MAX-MIN EQUATIONS WITH FUZZY SYMBOLS

by

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Abstract. We enlarge fuzzy equations to fuzzy symbols, already defined [1] as particular fuzzy sets whose membership functions operate between two complete and linearly ordered spaces. We solve equation $\tilde{A} \circledast X = \tilde{C}$ where $\tilde{A}$ e $\tilde{C}$ are fuzzy symbols, X is an unknown fuzzy set, and $\circledast$ is the extended max-min operations.

Key-words. Fuzzy symbols, extension principle, fuzzy equations.

1. Introduction.

The fuzzy symbols definition [1] raises by the exigence to assign the membership functions of a fuzzy set wherever it is hard to construct a homomorphism from a qualitative preference system into a quantitative preference system [2], [3].

We define fuzzy symbols as fuzzy sets whose membership function has a bell shaped graph [4] and his range is a complete, bounded and linearly ordered space. It makes possible to assign a geometric, logical or linguistic symbol to a state in the set of states of interest. In this way we can operate making use of not exactly definited expressions such as "approximately good", etc. In this work we want to enlarge to the fuzzy symbols the fuzzy equations studied by Sanchez [5] and [6] from theory and application [7] points of view. We study the existence of the solutions of the fuzzy max-min equations, then we search for the greatest solution.

2. Prerequisites.

Let D be a complete and linearly ordered space and S a complete, bounded and linearly ordered space whose greatest lower bound and least upper bound are respectively denotated by 0 and 1.

We define the <u>fuzzy symbol</u> $\tilde{A}$ as a normalized and convex fuzzy set whose membership function f operates from D into S, i.e.:

$$\max_{x \in D} f(x) = 1 ,$$

$$f(z) \geq \min (f(x), f(y))$$

$$\forall\ x, y \in D \quad e \quad x < y \quad e\ \forall z \in [x, y] .$$

So the f-graph is bell shaped, i.e.:

let a_f, b_f, c_f, d_f be elements of D such that: $c_f \quad a_f \quad b_f \quad d_f$, named <u>characteristic points</u> of $\tilde{A}$, the f function verifies the following conditions:

1) f is equal to 0 in $] \leftarrow, c_f]$,
2) f is strictly increasing in $[c, a]$,
3) f is equal to 1 in $[a_f, b_f]$,
4) f is strictly decreasing in $[b_f, d_f]$,
5) f is equal to 0 in $[d_f, \rightarrow[$.

Following the fuzzification principle exposed by Zadeh, the maximum ($\vee$) and minimum ($\wedge$) operations were extended to fuzzy symbols [1] ; later we shall use the sign $*$ to indicate both operations.

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy symbols whose membership functions are respectively f and g, we define the <u>fuzzy maximum of $\tilde{A}$ and $\tilde{B}$</u>, denoted $\tilde{A} \circledvee \tilde{B}$, a fuzzy set whose membership function h is

$$h(z) = \sup_{\substack{x, y \in D \\ x \vee y = z}} f(x) \wedge g(y) .$$

In a similar way we define <u>the fuzzy minimum of $\tilde{A}$ and $\tilde{B}$</u>, denoted $\tilde{A} \circledwedge \tilde{B}$, a fuzzy set whose membership function l is

$$l(z) = \sup_{\substack{x, y \in D \\ x \wedge y = z}} f(x) \wedge g(y).$$

The fuzzy symbols set is closed with respect to the above stated operations.

3. Consistence of max-min equations.

We consider the equation:

$$\tilde{A} \circledast X = \tilde{C} \qquad (1)$$

where $\tilde{A}$ and $\tilde{C}$ are fuzzy symbols and X is an unknown fuzzy set whose membership function operates from D into S. We intend to show the consistence of the equation (1) where $\circledast$ is the extended max-min operations (°).

Considering the characteristic points of fuzzy maximum $\tilde{A} \circledvee \tilde{B}$ verify the following relations:

$$a_h = a_f \vee a_g \qquad c_h = c_f \vee c_g$$

$$b_h = b_f \vee b_g \qquad d_h = d_f \vee d_g \qquad [1],$$

we have:

Theorem 1. The equation $\tilde{A} \circledvee X = \tilde{C}$ is consistent if and only if the characteristic points of fuzzy symbol $\tilde{C}$ are in the same order not lower of those of $\tilde{A}$.

(°) - In a similar way we can study the equation $X \circledast \tilde{A} = \tilde{C}$ for the commutative law of fuzzy maximum and fuzzy minimum.

Furthermore, considering the characteristic points of fuzzy minimum $\tilde{A} \circledwedge \tilde{B}$ verify the following relations:

$$a_l = a_f \wedge a_g \qquad c_l = c_f \wedge c_g$$
$$b_l = b_f \wedge b_g \qquad d_l = d_f \wedge d_g \quad [1],$$

we have:

Theorem 2. The equations $\tilde{A} \circledwedge X = \tilde{C}$ is consistent if and only if the characteristic points of fuzzy symbol $\tilde{C}$ are in the same order not greater of those of $\tilde{A}$.

When the consistence hypotheses are verified, the equation (1) always admits the fuzzy symbol $\tilde{C}$ as solution. In order to search for other solutions, we define the fuzzy set $\tilde{C} \circledast \tilde{A}$ by operator α used by Sanchez [6] for the solutions of fuzzy equations. The membership function k of fuzzy set $\tilde{C} \circledast \tilde{A}$ can be defined in the following way:

$$k(y) = \inf_{\substack{x, z \in D \\ x * y = z}} f(x) \, \alpha \, h(z)$$

where α is an operator of S such that:

$$f(x) \, \alpha \, h(z) = \begin{cases} 1 & \text{if } f(x) \leq h(z) \\ h(z) & \text{if } f(x) > h(z) \end{cases}.$$

So $f(x) \alpha h(z)$ is the greatest element χ in S such that

$$f(x) \wedge \chi \leq h(z).$$

4. The k-function graph.

Let k be the membership function of fuzzy set $\tilde{C} \circledvee \tilde{A}$ concerning the fuzzy maximum operation and $\tilde{A}$ and $\tilde{C}$ be fuzzy symbols whose membership functions respectively are f and h whose characteristic points verify the consistence hypotheses (Theorem 1).

Since:

$$z = x \vee y = \begin{cases} x & \text{if} \quad x \geq y \\ y & \text{if} \quad x < y \end{cases}$$

the k function can also be expressed as:

$$k(y) = \inf \left\{ \inf_{\substack{x \in D \\ x < y}} f(x) \alpha\, h(y), \ \inf_{\substack{x \in D \\ x \geq y}} f(x) \alpha\, h(x) \right\} .$$

Let

$$A(y) = \inf_{\substack{x \in D \\ x < y}} f(x) \alpha\, h(y)$$

one of the following propositions is true:

I) $\exists \ x^* < y : f(x^*) > h(y)$: then we have

$$A(y) = h(y).$$

II) $A(y) = 1$.

Let

$$B(y) = \inf_{\substack{x \in D \\ x \geq y}} f(x) \alpha\, h(x)$$

one of the folbwing propositions is true:

I) $x^* \geq y : f(x^*) > h(x^*)$, then we have

$$B(y) = \inf_{\substack{x \geq y \\ f(x) > h(x)}} h(x)$$

II) $B(y) = 1$.

We get:

Theorem 3. The membership k function of fuzzy set $\tilde{C}$ ⓥ $\tilde{A}$ verifies one of the following equalities:

1) if the characteristic points c_f of $\tilde{A}$ and c_h of $\tilde{C}$ are such as $c_f < c_h$ we have:

$$k(y) = h(y) \quad \forall \, y \in D ;$$

2) if $c_f = c_h$, named

$$\bar{y} = \inf\left\{ y \in D : y < a_f,\ f(y) > h(y) \right\}$$

we have

$$k(y) = h(\bar{y}) \quad \forall y \in \left] \leftarrow , \bar{y} \right]$$

$$k(y) = h(y) \quad \forall y \in \left] \bar{y}, \rightarrow \right[,$$

(figs. 1 and 2).

Now we consider k is the membership function of fuzzy set $\tilde{C} \circledwedge \tilde{A}$ concerning the fuzzy minimum operation where $\tilde{A}$ and $\tilde{C}$ are fuzzy symbols whose membership functions respectively are f and h and whose characteristic points verify the consistence hypotheses (Theorem 2).

We get:

Theorem 4. The membership k function of fuzzy set $\tilde{C} \circledwedge \tilde{A}$ verifies one of the following equalities:

1) if the characteristic points d_f of $\tilde{A}$ and d_l of $\tilde{C}$ are such as $d_l < d_f$ we have:

$$k(y) = l(y) \quad \forall y \in D;$$

2) if $d_l = d_f$, named

$$\bar{y} = \inf\left\{ y \in D : y > b_f,\ l(y) < f(y) \right\}$$

we have

$$k(y) = l(y) \quad \forall y \in \left] \leftarrow , \bar{y} \right[$$

$$k(y) = l(\bar{y}) \quad \forall y \in \left[\bar{y}, \rightarrow \right[.$$

5. Solutions of max-min equations.

The theorems 3 and 4 construct the graphs of k function related to fuzzy set $\tilde{C} \circledast \tilde{A}$, where the $*$ sign indicates both max-min operations, and led to the following

Theorem 5. For every two fuzzy symbols satisfying the consistence hypotheses of the equation (1) we get:

$$\tilde{A} \circledast (\tilde{C} \circledcirc \tilde{A}) = \tilde{C}.$$

In other words the fuzzy set $\tilde{C} \circledcirc \tilde{A}$ is a solution of equation (1). Moreover if a fuzzy set B is solution of equation (1), in accord to the general fuzzy equations of extended operations, we have

$$B \subseteq \tilde{C} \circledcirc \tilde{A} .$$

Then fuzzy set $\tilde{C} \circledcirc \tilde{A}$ is the greatest solution of equation (1). At last, being the extended operations distributive over the unione, i.e.:

$$A \circledast (B_1 \cup B_2) = (A \circledast B_1) \cup (A \circledast B_2)$$

we deduce that the set of solutions of equation (1) is an upper semi-lattice.

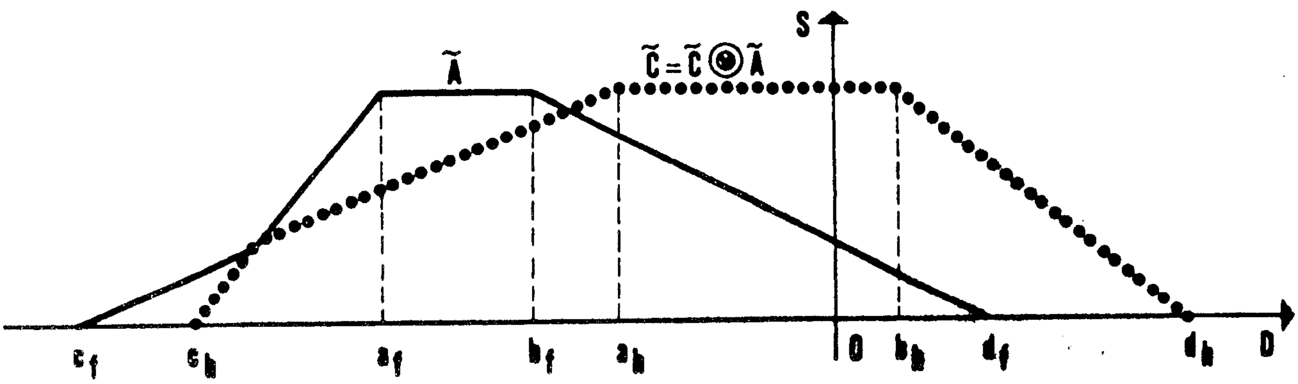

fig. 1

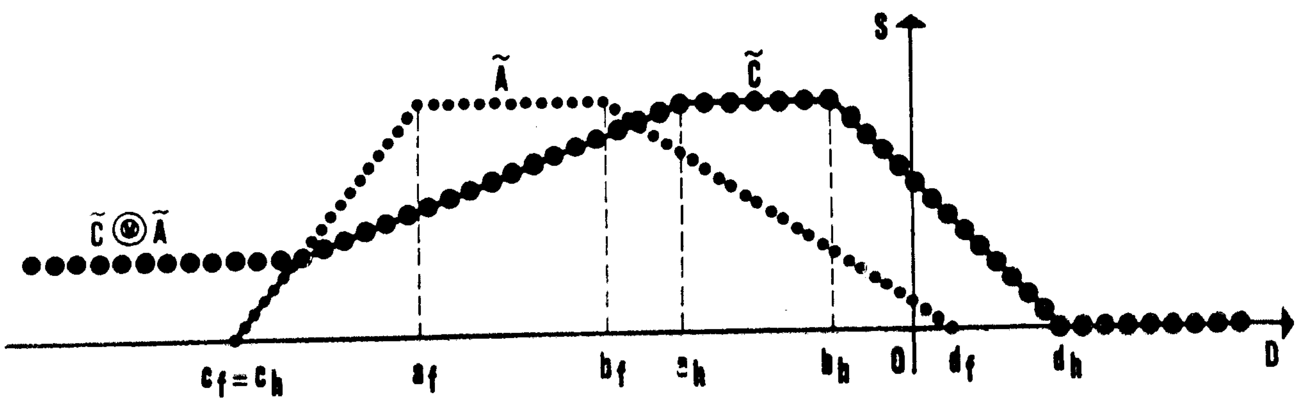

fig. 2

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