GENERALIZATION OF SOME RESULTS OF ELIE SANCHEZ

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Abstract Recently E. Sanchez has given a necessary and sufficient condition for the existence of a solution of a fuzzy equation $A * X = C$, with $*$ the extension of a given binary operation in the universe of discourse.

Using many valued logic we generalize this result and get a characterization also of the cases if $A * X = C$ has only nearly a solution.

1. Introduction

Just recently E. SANCHEZ /5/ was considering the problem of solving set equations $A * X = B$ for fuzzy sets A,B where the operation $*$ extends an operation in the common universe of discourse of A,X. The problem is relevant e.g. for the arithmetics of fuzzy numbers.

Using as in earlier papers /2/,/3/ essentially the language and results of many valued logics we will sketch here how these means are helpful to give his results a greater generality and power.

The present results have been found during a summer school in Primorsko/Bulgaria. The author kindly acknowledges his invitation and the encouraging and fruitful conditions of this meeting, realized by the organizers.

2. Logical Notation

Mainly the same notation as in /3/ is used. For many valued logic, especially, we do not distinguish the connectives and their corresponding truth functions. The set of generalized truth values is the real interval $[0,1]$; the connectives we use are conjunctions described by

$$s \wedge_1 t = \min(s,t) \quad ,$$
$$s \wedge_2 t = \max(0, s+t-1) \quad ,$$

and implications described by

$$s \rightarrow_1 t = \min(1, 1-s+t) \quad ,$$
$$s \rightarrow_2 t = \begin{cases} 1, & \text{if } s \leq t \\ t & \text{otherwise} . \end{cases}$$

Of course, this implication $\rightarrow_2$ is nothing else than the α-operation of E. SANCHEZ /4/,/5/. But, $\rightarrow_2$ is also that many valued implication operator which was used by K. GÖDEL /1/ for proving that intuitionistic propositional logic does not have a finite adequate matrix. Therefore we prefer to denote this operation as an implication.

We write $\models H$ to indicate, that formula H of our many valued language always has truth value 1; and generally we denote by $[H]$ the truth value of formula H.

For quantification we suppose that generalization $\forall$ corresponds to taking the infimum of truth values:

$$[\forall x\, H(x)] = \inf_{x \in \underline{U}} [H(x)] \quad ,$$

$\underline{U}$ the common universe of discourse for all our fuzzy sets, and that existential quantification $\exists$ corresponds to taking the supremum of truth values.

Obviously, with our notation

$$\models H_1 \rightarrow H_2 \qquad \text{iff} \qquad [H_1] \leq [H_2] \qquad (*)$$

for $\rightarrow$ any one of $\rightarrow_1$, $\rightarrow_2$. And with biimplication defined by

$$H_1 \leftrightarrow H_2 =_{def} (H_1 \rightarrow H_2) \wedge_1 (H_2 \rightarrow H_1) \quad ,$$

again $\rightarrow$ any one of $\rightarrow_1$, $\rightarrow_2$ (but the same in both occurences), we get

$$\models H_1 \leftrightarrow H_2 \qquad \text{iff} \qquad [H_1] = [H_2] \, . \qquad (+)$$

In the following, sometimes the choice of $\rightarrow_1$, $\rightarrow_2$ or also of $\leftrightarrow_1$, $\leftrightarrow_2$ is inessential. Then we will write $\longrightarrow$ resp. $\leftrightarrow$ only. (In all these cases for $\rightarrow$, $\leftrightarrow$ any connective may be chosen that meets condition $(*)$, $(+)$ respectively, but it is not necessary to take this more general point of view here.)

3. Set Theoretical Notation

Generalized membership within a fuzzy set we describe by our many valued membership predicate ε putting

$$[x \,\varepsilon\, A] =_{def} \mu_A(x) \quad .$$

The fact that a fuzzy set is uniquely characterized by its membership function we use to introduce class terms. Having given a fuzzy set A and a formula $H(x)$ of our many valued language such that

$$\mu_A(x) = [H(x)] \quad \text{for all } x \in \underline{U} ,$$

$\underline{U}$ the universe of discourse, we define class terms by

$$\{x \parallel H(x)\} =_{def} A .$$

The usual intersection of fuzzy sets A,B, denoted: $A \cap_1 B$, now can be characterized through

$$A \cap_1 B = \{x \parallel x \in A \wedge_1 x \in B\} ,$$

the cartesian product e.g. through

$$A \times_1 B = \{(x,y) \parallel x \in A \wedge_1 y \in B\} .$$

More generally, for every binary function f on the universe of discourse $\underline{U}$ we write ($\wedge$ any one of $\wedge_1$, $\wedge_2$)

$$\{f(x,y) \parallel x \in A \wedge y \in B\}$$

for the set

$$\{z \parallel \exists x \exists y((x \in A \wedge y \in B) \wedge_1 z \doteq f(x,y))\} ,$$

putting additionally

$$[u \doteq v] = \begin{cases} 1, & \text{if } u = v \\ 0 & \text{otherwise} . \end{cases}$$

With this notation, the <u>extension principle</u> of L. ZADEH /6/ becomes:

Given an operation $*$ within $\underline{U}$, extend it to the fuzzy subsets of $\underline{U}$ by putting

$$A * B = \{a * b \parallel a \in A \wedge_1 b \in B\} .$$

Finally, we generalize the usual two-valued inclusion relation for fuzzy sets to many valued ones defining

$$A \subseteq_i B =_{def} \forall x(x \in A \rightarrow_i x \in B) ,$$

with $i = 1,2$. Again we write simply $A \subseteq B$ if it does not matter if we choose $\subseteq_1$ or $\subseteq_2$. And it is easy to see that

$$\models A \subseteq_2 B \longrightarrow A \subseteq_1 B \quad ,$$

$$\models A \subseteq_2 B \quad \text{iff} \quad \models A \subseteq_1 B \quad ,$$

and

$$\models A \subseteq_2 B \wedge_1 B \subseteq_2 C \longrightarrow A \subseteq_2 C \quad .$$

4. Fuzzy Equations

Now we are ready to describe the results. We follow the way of presentation in /5/. The proofs will be given elsewhere. With some simple facts of our many valued logic, which will not be explained here, they are almost straightforward.

Westart from a binary operation $*$ within our universe of discourse $\underline{U}$, and extend it to fuzzy sets over $\underline{U}$ in accordance with the extension principle. Immediately we get

Corollary 1. For all $x \in \underline{U}$

$$\models x \varepsilon A * B \longleftrightarrow \exists a \exists b (a \varepsilon A \wedge_1 b \varepsilon B \wedge_1 x \doteq a * b) \quad .$$

Proposition 1.

(i) $\models A \subseteq B \longrightarrow A * C \subseteq B * C$,

(ii) $\models A \subseteq B \longrightarrow C * A \subseteq C * B$.

Corollary 2.

$$\models A \subseteq C \wedge_2 B \subseteq D \longrightarrow A * B \subseteq C * D \quad .$$

Proposition 2.

(i) $A * (\bigcup_{j \in J} B_j) = \bigcup_{j \in J} (A * B_j)$,

(ii) $(\bigcup_{j \in J} A_j) * B = \bigcup_{j \in J} (A_j * B)$.

From this, using the support $|B|$ of fuzzy set B, we get

Corollary 3.

$$A * B = \bigcup_{b \in |B|} (\{ x * b \,\|\, x \varepsilon A \wedge_1 b \varepsilon B \}) \quad .$$

Corollary 4. If $*$ has in U an unique left inverse $\boxast$ such that: $z = x*y$ iff $x = z \boxast y$, then

$$A * B = \bigcup_{b \in |B|} (\{z \,\|\, z \boxast b \in A \wedge_1 b \in B\}) .$$

Definition 1 (cf. /4/,/5/).

$$C \tilde{*} A =_{def} \{y \,\|\, \forall x (x \in A \rightarrow_2 x*y \in C)\} .$$

Proposition 3.

$$\models C \subseteq_2 D \longrightarrow C \tilde{*} A \subseteq_2 D \tilde{*} A .$$

Proposition 4.

(i) $\models A * (C \tilde{*} A) \subseteq C$,

(ii) $\models B \subseteq (A * B) \tilde{*} A$.

Proposition 5.

(i) $\models A * B \subseteq_2 C \longrightarrow B \subseteq_2 C \tilde{*} A$,

(ii) $\models B \subseteq C \tilde{*} A \longrightarrow A * B \subseteq C$.

Corollary 5.

$$\models A * B \subseteq_2 C \longleftrightarrow B \subseteq_2 C \tilde{*} A .$$

Definition 2.

$$A \equiv_2 B =_{def} A \subseteq_2 B \wedge_1 B \subseteq_2 A .$$

Corollary 6.

$$\models A \equiv_2 B \quad \text{iff} \quad A = B .$$

THEOREM.

$$\models \exists X (A * X \equiv_2 C) \longleftrightarrow A * (C \tilde{*} A) \equiv_2 C .$$

Corollary 7.

(i) $\models \exists X (A * X \equiv_2 C)$ iff $A * (C \tilde{*} A) = C$,

(ii) $\models \exists X (A * X \equiv_2 C)$ iff $A * X = C$ has a solution.

This last result, Corollary 7, contains the main theorem of /5/. The generalization we proved in our theorem now gives not only: $A * X = C$ has a solution iff $A * (C \tilde{*} A) = C$, but furthermore: the degree of equality (in sense of $\equiv_2$) of $A * (C \tilde{*} A)$ and C is the same as the degree of existence of a solution of $A * X \equiv_2 C$.

Or, to put it more loosely: if $A * (C \tilde{*} A)$ and C are nearly equal then $A * X \equiv_2 C$ is (to the same degree) nearly solvable, and vice versa.

References

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