A NEW ALGORITHEM FOR COMPUTING THE TRANSITIVE CLOSURES OF FUZZY PROXIMITY RELATION MATRICES

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Abstract

In this paper, We approach other a handy way of computation the transitive closure of a finite fuzzy proximity relation.

INTRODUCTION

By a proximity relation means a reflexive, symmetrical fuzzy relation. By a similarity relation means a reflexive, symmetrical, max-min transitive fuzzy relation.

As everyone knows, a proximity relation on finite universes can be represented as a matrix R:

$$R = \begin{bmatrix} 1 & r_{12} & \cdots & r_{1n} \\ r_{12} & 1 & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{1n} & r_{2n} & \cdots & 1 \end{bmatrix}$$

write

$$R^* = R \cup R^2 \cup \ldots \cup R^m \cup \ldots,$$

$$= \begin{bmatrix} 1 & r^*_{12} & \cdots & r^*_{1n} \\ r^*_{12} & 1 & \cdots & r^*_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r^*_{1n} & r^*_{2n} & \cdots & 1 \end{bmatrix}$$

then, R^* is the transitive closure of R.

Because of R is symmetrical, we only considers the elements of R above the diogonal, and arranges these elements in the order from large to small: and denotes the prior P elements as: $S_1, S_2, \dots, S_p$,

where $p = \left[\frac{n}{2}\right]$ denotes the maximal round number of $\frac{n}{2}$;

$$S_i \geq S_j, \quad \text{for } i < j.$$

Let $T = \{1, 2, \dots, P\}$, and $S_t = r_{i_t j_t}$ and

$$B_t = \{i_t, j_t\}, \quad t \in T.$$

<u>Definition 1</u> Given S, Let $G = \bigcup_{t \in S \subset T} B_t$

G is called associated set iff:

1, For all $t \in S \subset T$, there are $m \in B_t$, and $s \in S \setminus \{t\}$, such that $n \in B_s$ and $m = n$.

2, $G \cap \left(\bigcup_{\alpha \in T \setminus S} B_\alpha \right) = \emptyset$

Especially, if for all $s \neq t \in T$, $B_s \cap B_t = \emptyset$, then every B_t is called still associated set.

The number of associated set is called eigen of R, denoted as e.

<u>Definition 2</u> Let G_i $(i = 1, 2, \dots, e)$ be associated sets. Let

$$B_1 = \{a_i \mid i = 1, 2, \dots, m;\ a_i < a_j,\ i < j\}$$

$$B_2 = \{b_i \mid i = 1, 2, \dots, n;\ b_i < b_j,\ i < j\}$$

$$\dots \quad \dots$$

$$B_e = \{c_i \mid i = 1, 2, \dots, q;\ c_i < c_j,\ i < j\}$$

$$D = \{d_i \mid i = 1, 2, \dots, t;\ d_i < d_j,\ i < j\} = \{1, 2, \dots, n\} \setminus \bigcup_{k=1}^{e} B_k.$$

and

$$S_1 = \begin{bmatrix} 1 & r_{a_1a_2} & \cdots & r_{a_1a_m} \\ r_{a_1a_2} & 1 & \cdots & r_{a_2a_m} \\ \cdots & \cdots & \cdots & \cdots \\ r_{a_1a_m} & r_{a_2a_m} & \cdots & 1 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 1 & r_{b_1b_2} & \cdots & r_{b_1b_n} \\ r_{b_1b_2} & 1 & \cdots & r_{b_2b_n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{b_1b_n} & r_{b_2b_n} & \cdots & 1 \end{bmatrix}$$

... ...

$$S_e = \begin{bmatrix} 1 & r_{c_1c_2} & \cdots & r_{c_1c_q} \\ r_{c_1c_2} & 1 & \cdots & r_{c_2c_q} \\ \cdots & \cdots & \cdots & \cdots \\ r_{c_1c_q} & r_{c_2c_q} & \cdots & 1 \end{bmatrix}$$

We call the matrices S_1, S_2, ..., S_e. associated submatrices.

<u>Definition 3</u> Let

$$A_{ii} = S_i \quad (i \leq e)$$
$$A_{ii} = 1 \quad (i > e)$$

$$A_{12} = \begin{bmatrix} r_{a_1b_1} & r_{a_1b_2} & \cdots & r_{a_1b_n} \\ r_{a_2b_1} & r_{a_2b_2} & \cdots & r_{a_2b_n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{a_mb_1} & r_{a_mb_2} & \cdots & r_{a_mb_n} \end{bmatrix}$$

... ...

$$A_{11} = \begin{bmatrix} r_{a_1c_1} & r_{a_1c_2} & \cdots & r_{a_1c_q} \\ r_{a_2c_1} & r_{a_2c_2} & \cdots & r_{a_2c_q} \\ \cdots & \cdots & \cdots & \cdots \\ r_{a_mc_1} & r_{a_mc_2} & \cdots & r_{a_mc_q} \end{bmatrix}, \quad A_{1,e+1} = \begin{bmatrix} r_{a_1d_1} \\ r_{a_2d_1} \\ \vdots \\ r_{a_md_1} \end{bmatrix}, \quad \cdots\ \cdots,$$

$$A_{1,e+\alpha} = \begin{bmatrix} r_{a_1d_t} \\ r_{a_2d_t} \\ \vdots \\ r_{a_md_t} \end{bmatrix}, \quad \cdots, A_{2e} = \begin{bmatrix} r_{b_1c_1} & \cdots & r_{b_1c_q} \\ \cdots & \cdots & \cdots \\ r_{b_nc_1} & \cdots & r_{b_nc_q} \end{bmatrix}, \quad A_{2,e+1} = \begin{bmatrix} r_{b_1d_1} \\ r_{b_2d_1} \\ \vdots \\ r_{b_nd_1} \end{bmatrix}$$

$$\cdots, A_{2,e+\alpha} = \begin{bmatrix} r_{b_1d_t} \\ r_{b_2d_t} \\ \vdots \\ r_{b_2d_t} \end{bmatrix} \cdots, A_{e,e+1} = \begin{bmatrix} r_{c_1d_1} \\ r_{c_2d_1} \\ \vdots \\ r_{c_qd_1} \end{bmatrix} \cdots \quad A_{e,e+\alpha} = \begin{bmatrix} r_{c_1d_t} \\ r_{c_2d_t} \\ \\ r_{c_qd_t} \end{bmatrix},$$

$\cdots, \quad A_{e+i,\, e+j} = r_{d_id_j}$

and $A_{ji} = A^T_{ij}$ (A^T_{ij} denotes transpose of submatrix A_{ij}).

Let U_{ij} denote the maximal element of A_{ij}. We call matrix

$$V = (U_{ij})_{(e+\alpha)\times(e+\alpha)}$$

auxiliary matrix. where n=P +α + 1.

Definition 4 Suppose R transposed to T by exchanging the ith row with the jth row of R, and again T transposed to S by exchanging the ith column with the jth column of T; here we call R and S <u>equivalent</u>, and denote it as S ~ R

Proposition 1 Let R and S be fuzzy proximity matrix, then

$$R \sim S \Longrightarrow R^* \sim S^*$$

Proof Suppose

$$\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1i} & \cdots & r_{1j} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2i} & \cdots & r_{2j} & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{i1} & r_{i2} & \cdots & r_{ii} & \cdots & r_{ij} & \cdots & r_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{j1} & r_{j2} & \cdots & r_{ji} & \cdots & r_{jj} & \cdots & r_{jn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{n2} & \cdots & r_{ni} & \cdots & r_{nj} & \cdots & r_{nn} \end{bmatrix} = R ,$$

and

$$S = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1j} & \cdots & r_{1i} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2j} & \cdots & r_{2i} & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{j1} & r_{j2} & \cdots & r_{jj} & \cdots & r_{ji} & \cdots & r_{jn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{i1} & r_{i2} & \cdots & r_{ij} & \cdots & r_{ii} & \cdots & r_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{n2} & \cdots & r_{nj} & \cdots & r_{ni} & \cdots & r_{nn} \end{bmatrix}$$

then the elements of S^2 are determined by the multiplication rule:

$$r_{ij}^{(2)} = \bigvee_{k=1}^{n} (r_{ik} \wedge r_{kj}).$$

Therefore, $R^2 \sim S^2$, and so, $R^m \sim S^m$, consequently

$$R^* \sim S^* .$$

For convenience, we denote $A_{ij} = (a_{kl})_{m \times m}$,

$$T_{ij} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \\ 1 & 1 & \dots & 1 \end{bmatrix}_{m \times n}, \qquad B_{ij} = \begin{bmatrix} \alpha & \alpha & \cdots & \alpha \\ \cdots & \cdots & \cdots & \\ \alpha & \alpha & \cdots & \alpha \end{bmatrix}_{m \times m}$$

<u>Proposition 2.</u> If $e = 1$, then $R^* = (A^*_{ij})$, Where

$$A^*_{ij} = \begin{cases} S^*, & i=j=1; \\ U^*_{1j} B_{1j}, & i=1,\ j \neq 1; \\ U^*_{1j} B_{j1}, & i \neq 1,\ j=1; \\ U^*_{ij}, & i \neq 1,\ j \neq 1 . \end{cases}$$

If $e \neq 1$, then

$$A^*_{ij} = \begin{cases} S^*_i, & i=j \leq e; \\ S^{*}_{ij}, & i=j > e; \\ U^*_{ij} B_{ij}, & i \neq j. \end{cases}$$

Proof. If $e=1$, then $B = \prod_{t \in T} B_t$.

We only proof the following condition:

$$R = \begin{bmatrix} 1 & s_1 & s_2 & \cdots & s_p & r_{1,p+2} & \cdots & r_{1n} \\ s_1 & 1 & \cdots & \cdots & r_{2,p+1} & r_{2,p+2} & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ s_p & r_{2,p+1} & \cdots & \cdots & 1 & r_{p+1,p+2} & \cdots & r_{p+1,n} \\ r_{1,p+2} & r_{2,p+2} & \cdots & \cdots & r_{p+1,p+2} & 1 & \cdots & r_{p+2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{1n} & r_{2n} & \cdots & \cdots & r_{p+1,n} & r_{p+2,n} & \cdots & 1 \end{bmatrix}$$

We can proof the others conditions in the same way.

$$A_{11} = S = \begin{bmatrix} 1 & s_1 & \cdots & s_p \\ s_1 & 1 & \cdots & r_{2,p+1} \\ \cdots & \cdots & \cdots & \cdots \\ s_p & r_{2,p+1} & \cdots & 1 \end{bmatrix} \qquad A_{12} = \begin{bmatrix} r_{1,p+2} \\ r_{2,p+2} \\ \cdots \\ r_{p+1,p+2} \end{bmatrix} \cdots ,$$

$$A_{1,\alpha+1} = \begin{bmatrix} r_{1n} \\ r_{2n} \\ \vdots \\ r_{p+1,n} \end{bmatrix}, \qquad A_{22}=1, \ \ldots, \ A_{\alpha+1,\alpha+1}=1.$$

$$(n = p + \alpha + 1).$$

According to the induction and multiplication rule of matrix, we can proof:

$$A_{ij}^{(m+1)} = \bigcup_{k_1,\ldots,k_m=1}^{\alpha+1} (A_{ik_1} \circ A_{k_1k_2} \circ \ldots \circ A_{k_mj}) \qquad (1)$$

Thereupon, we further may proof:

$$A_{ij}^{(n-1)} = \bigcup_{k=1}^{\alpha+1} (A_{ii}^{p-1} \circ A_{ik} \circ A_{kk}^{\alpha-1} \circ A_{kj}) \qquad (2)$$

from the formula (1), we obtain

$$A_{ij}^{(n-1)} = \bigcup_{k_1\ldots k_{n-2}=1}^{\alpha+1} (A_{ik_1} \circ A_{k_1k_2} \circ \ldots \circ A_{k_{n-2}j}) ,$$

clearly we have

$$\bigcup_{k=1}^{\alpha+1} (A_{ii}^{p-1} \circ A_{ik} \circ A_{kk}^{\alpha-1} \circ A_{kj}) \subseteq \bigcup_{k_1\ldots k_{n-2}=1}^{\alpha+1} (A_{ik_1} \circ \ldots \circ A_{k_{n-2}j})$$

But, because $A_{ii}=S$ or 1, for any $k_1, \ldots, k_{n_2}$, there is

$$A_{ii}^{p-1} \circ A_{ik_p} \circ A_{k_pk_p}^{\alpha-1} \circ A_{k_pj} \supseteq A_{ik_1} \circ A_{k_1k_2} \circ \ldots \circ A_{k_{n-2}j} ,$$

therefore

$$\bigcup_{k_p=1}^{\alpha+1} (A_{ii}^{p-1} \circ A_{ik_p} \circ A_{k_pk_p}^{\alpha-1} \circ A_{k_pj}) \supseteq \bigcup_{k_1\ldots k_{n-2}=1}^{\alpha+1} (A_{ik_1} \circ \ldots \circ A_{k_{n-2}j}).$$

If $e \neq 1$, then $1 < e \leq p$. we only proof condition on the $e=p$ and $n=2p$. By the proposition 1, we only need proofing following condition:

$$\begin{bmatrix} 1 & S_1 & r_{13} & r_{14} & \cdots & r_{1n-1} & r_{1n} \\ S_1 & 1 & r_{23} & r_{24} & \cdots & r_{2n-1} & r_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{1,n-1} & r_{2,n-1} & r_{3,n-1} & r_{4,n-1} & \cdots & 1 & S_p \\ r_{1n} & r_{2n} & r_{3n} & r_{4n} & \cdots & S_p & 1 \end{bmatrix}$$

[illegible] proof:

$$A_{ij}^{(n-1)} = \bigcup_{k=1}^{p} (A_{ii}^{p-2} \circ A_{ik} \circ A_{kk}^{p-2} \circ A_{kj} \circ A_{jj}).$$

[illegible] the proposition is still a genuine.

[illegible] 3. The transitive closure of any finite fuzzy [illegible] relation is obtained by means the closure of matrices of order 2, 3.

Example Let

$$R = \begin{bmatrix} 1 & 0.9 & 0.1 & 0.6 & 0.2 & 0.1 & 0.3 & 0.4 \\ 0.9 & 1 & 0.3 & 0.5 & 0.5 & 0.8 & 0.2 & 0.7 \\ 0.1 & 0.3 & 1 & 0.2 & 0.2 & 0.4 & 0.3 & 0.1 \\ 0.6 & 0.5 & 0.2 & 1 & 0.1 & 0.3 & 0.2 & 0.4 \\ 0.2 & 0.5 & 0.2 & 0.1 & 1 & 0.2 & 0.1 & 0.3 \\ 0.1 & 0.8 & 0.4 & 0.3 & 0.2 & 1 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.3 & 0.2 & 0.1 & 0.3 & 1 & 0.1 \\ 0.4 & 0.7 & 0.1 & 0.4 & 0.3 & 0.2 & 0.1 & 1 \end{bmatrix}$$

[illegible] $\left[\frac{8}{2}\right]=4$; $S_1=r_{12}$, $S_2=r_{26}$, $S_3=r_{28}$, $S_4=r_{14}$; $e=1$

[illegible]$=\{[illegible], 2, 4, 6, 8\}$, $D=\{3, 5, 7\}$;

[illegible] $A_{24}=0.3$, $A_{33}=1$, $A_{34}=0.1$, $A_{44}=1$,

$A_{22}=1$, $A_{23}=0.2$

$$S = \begin{bmatrix} 1 & 0.9 & 0.6 & 0.1 & 0.4 \\ 0.9 & 1 & 0.5 & 0.8 & 0.7 \\ 0.6 & 0.5 & 1 & 0.3 & 0.4 \\ 0.1 & 0.8 & 0.3 & 1 & 0.2 \\ 0.4 & 0.7 & 0.4 & 0.2 & 1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.2 \\ 0.4 \\ 0.1 \end{bmatrix}$$

$$A_{13} = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.1 \\ 0.2 \\ 0.3 \end{bmatrix} \quad A_{14} = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.2 \\ 0.3 \\ 0.1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 0.4 & 0.5 & 0.3 \\ 0.4 & 1 & 0.2 & 0.3 \\ 0.5 & 0.2 & 1 & 0.1 \\ 0.3 & 0.3 & 0.1 & 1 \end{bmatrix}$$

Because of S and U are not matrices of order 2,3, we [illegible]ceed with step 1,2. First of all, as for S:

$[illegible]\left[\frac{5}{2}\right] = 2;\quad S_1 = r_{12},\quad S_2 = r_{14};\quad e=1,\quad G=\{1,2,4\},\quad D=\{3,5\};$

$$[illegible] = \begin{bmatrix} 1 & 0.9 & 0.1 \\ 0.9 & 1 & 0.8 \\ 0.1 & 0.8 & 1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.6 \\ 0.5 \\ 0.3 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 0.4 \\ 0.7 \\ 0.2 \end{bmatrix},$$

$[illegible] \quad A_{23}=0.4, \quad A_{33}=1,$

$$[illegible] = \begin{bmatrix} 1 & 0.6 & 0.7 \\ 0.6 & 1 & 0.4 \\ 0.7 & 0.4 & 1 \end{bmatrix}$$

Where, S(S) denotes associated submatrix of S. Similaiy U(S) denotes auxiliary matrix of S.

[illegible] obtain that

$$S^* = \begin{bmatrix} 1 & 0.9 & 0.6 & 0.8 & 0.7 \\ 0.9 & 1 & 0.6 & 0.8 & 0.7 \\ 0.6 & 0.6 & 1 & 0.6 & 0.6 \\ 0.8 & 0.8 & 0.6 & 1 & 0.7 \\ 0.7 & 0.7 & 0.6 & 0.7 & 1 \end{bmatrix} \quad U^* = \begin{bmatrix} 1 & 0.4 & 0.5 & 0.3 \\ 0.4 & 1 & 0.4 & 0.3 \\ 0.5 & 0.4 & 1 & 0.3 \\ 0.3 & 0.3 & 0.3 & 1 \end{bmatrix}$$

At last, according to S* and U*, we obtain consequence:

$$R^* = \begin{bmatrix} 1 & 0.9 & 0.4 & 0.6 & 0.5 & 0.8 & 0.3 & 0.7 \\ 0.9 & 1 & 0.4 & 0.6 & 0.5 & 0.8 & 0.3 & 0.7 \\ 0.4 & 0.4 & 1 & 0.4 & 0.4 & 0.4 & 0.3 & 0.4 \\ 0.6 & 0.6 & 0.4 & 1 & 0.5 & 0.6 & 0.3 & 0.6 \\ 0.5 & 0.5 & 0.4 & 0.5 & 1 & 0.5 & 0.3 & 0.5 \\ 0.8 & 0.8 & 0.4 & 0.6 & 0.5 & 1 & 0.3 & 0.7 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 1 & 0.3 \\ 0.7 & 0.7 & 0.4 & 0.6 & 0.5 & 0.7 & 0.3 & 1 \end{bmatrix}$$

The writer sincere thanks is also due to professor peizhuang Wang of Beijing Normal University.

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