SOLUTION TO A PROBLEM OF KIM AND ROUSH *

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Abstract

We solve the second open problem of K.H.KIM and F.W.ROUSH [1] in this paper.

Key words: Fuzzy matrices, Permutation matrices.

The second open problem K.H.KIM and F.W.ROUSH put forward in their artical [1] is stated as follows:

For an idempotent fuzzy matrix E does there exist a permutation matrix P such that $F=PEP^T$ satisfies $f_{ij} \geq f_{ji}$ for $i > j$?

We shall prove the answer is 'yes'. In fact, we shall prove this fact is true for any transitive fuzzy matrix over a lattice.

Let $\{L, \wedge, \vee, 0, 1\}$ be a completely distributive lattice such that $a \wedge b$ equals a or b (i.e. totally ordered). A matrix over L is called an L-fuzzy matrix, or fuzzy matrix for short. Let $L^{n\times m}$ denote the set of all $n\times m$ L-fuzzy matrices.

An $n\times n$ fuzzy matrix D is called a transitive fuzzy matrix if $D^2 \leq D$.

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By a short computation, we have the following result.

LEMMA: If $A = \begin{pmatrix} a_{11} & a_1 \\ a_2 & A_1 \end{pmatrix} = \begin{pmatrix} A_1' & a_1' \\ a_2' & a_{nn} \end{pmatrix} \in L^{n\times n}$

is a transitive fuzzy matrix, then A_1, A_1' and PAP^T are transitive fuzzy matrices for any permutation matrix P, where $a_1^T, a_2, a_1', a_2'^T \in L^{n\times 1}$, $A_1, A_1' \in L^{(n-1)\times(n-1)}$.

The main result is

THEOREM: For a transitive fuzzy matrix D there exists a permutation matrix P such that $F = PDP^T$ satisfies $f_{ij} \geq f_{ji}$ for $i > j$.

Proof: Using the inductive method on the order n of D.

$n = 2$, let $D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$.

If $d_{21} \geq d_{12}$, let $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

If $d_{21} < d_{12}$, let $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

It is straightforward to verify that the permutation matrix P is desired.

For the case of $n \geq 3$, put $D = \begin{pmatrix} d_{11} & d_1 \\ d_2 & D_1 \end{pmatrix}$.

By the LEMMA and inductive hypothesis, there exists a permutation matrix P_1 such that $P_1 D_1 P_1^T = (\bar{e}_{ij})$ satisfies $\bar{e}_{ij} \geq \bar{e}_{ji}$ for $i > j$.

Suppose $E = \begin{pmatrix} 1 & 0 \\ 0 & P_1 \end{pmatrix} D \begin{pmatrix} 1 & 0 \\ 0 & P_1^T \end{pmatrix} = \begin{pmatrix} d_{11} & d_1 P_1^T \\ P_1 d_2 & P_1 D_1 P_1^T \end{pmatrix} = (e_{ij})$.

Then $e_{ij} \geq e_{ji}$ for $1 < j \leq i \leq n$.

(1) If $e_{1n} \leq e_{n1}$, then $E = \begin{pmatrix} E_1 & e_1 \\ e_2 & e_{nn} \end{pmatrix}$

satisfies $e_2 \geq e_1^T$. By inductive hypothesis, we have a permutation matrix P_2 such that

$$F = \begin{pmatrix} P_2 & 0 \\ 0 & 1 \end{pmatrix} E \begin{pmatrix} P_2^T & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} P_2E_1P_2^T & P_2e_1 \\ e_2P_2^T & e_{nn} \end{pmatrix} = (f_{ij})$$

satisfies $f_{ij} \geq f_{ji}$ for $i > j$, since $e_2P_2^T \geq (P_2e_1)^T$. That is to say,

$$P = \begin{pmatrix} P_2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & P_1 \end{pmatrix}$$

is the desired permutation matrix for D.

(2) If $e_{1n} > e_{n1}$, then $e_{1k} \geq e_{k1}$ must hold for $1 < k < n$. If not, i.e., $e_{1k} < e_{k1}$ for some $1 < k < n$. By assumption and the LEMMA, we have $e_{nk} \geq e_{kn}$ and $E^2 \leq E$. So that

$$e_{ij} \geq \bigvee_{m=1,2,\cdots,n} (e_{im} \wedge e_{mj}) \geq e_{im} \wedge e_{mj} .$$

Suppose $e_{k1} \geq e_{1n}$. Then

$$e_{nk} \geq e_{kn} \geq e_{k1} \wedge e_{1n} = e_{1n}$$

and

$$e_{k1} > e_{1k} \geq e_{1n} \wedge e_{nk} = e_{1n} .$$

Hence, we have two contradictory equalities:

$$e_{1n} > e_{n1} \geq e_{nk} \wedge e_{k1} = e_{nk} \geq e_{kn}$$

$$e_{kn} \geq e_{k1} \wedge e_{1n} = e_{1n} .$$

Suppose $e_{k1} < e_{1n}$. Then

$$e_{nk} \geq e_{kn} \geq e_{k1} \wedge e_{1n} = e_{k1} > e_{1k} ,$$

we have acontradiction also

$$e_{1k} \geq e_{1n} \wedge e_{nk} = e_{1n} > e_{k1} .$$

Therefore $e_{k1} \leq e_{1k}$ holds for all $k = 2,3,\cdots,n-1$.

Let

$$P_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Then
$$P_n E P_n^T = \begin{pmatrix} E_2' & e_1' \\ e_2' & e_{11} \end{pmatrix}$$
satisfies $e_2'^T \geq e_1'$. By inductive hypothesis, there exists a permutation matrix P_2 such that
$$F = \begin{pmatrix} P_2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} E_2' & e_1' \\ e_2' & e_{11} \end{pmatrix}\begin{pmatrix} P_2^T & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} P_2 E_2' P_2^T & P_2 e_1' \\ e_2' P_2^T & e_{11} \end{pmatrix} = (f_{ij})$$
satisfies $f_{ij} \geq f_{ji}$ for $i>j$, since $e_2' P_2^T \geq (P_2 e_1')^T$. Thus
$$P = \begin{pmatrix} P_2 & 0 \\ 0 & 1 \end{pmatrix} P_n \begin{pmatrix} 1 & 0 \\ 0 & P_1 \end{pmatrix}$$
as desired.

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Reference

[1] K.H.KIM and F.W.ROUSH, Generalized Fuzzy Matrices, Fuzzy Sets and Systems 4(1980)293--315.