QUASI-METRIC SPACE AND FUZZY NUMBER

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As stated in [2], [3] and [4], in soft mathematics the triangle inequality of common metric space does not holds. Thus the application of common metric space to these subjects is restricted. To meet the needs of new problems we introduce the concepts of quasi-metric space and pseudo-metric space. Some properties of these concepts are discussed. At the same time we try to find some relations between quasi-metric space, pseudo-metric space and common metric space. On the basis of above concepts and consequences we difine a new space, the "F-metric space". At last we discuss the convex fuzzy sets and fuzzy numbers, which is useful in fuzzy mathematics such as fuzzy programming.

§1 Quasi-metric function and quasi-metric space

<u>Definition 1</u> Suppose $(X, \leq)$ be a poset. Define a mapping as follows:

$$\rho : X \times X \to [0, \infty).$$

The ρ is called a quasi-metric function on $(X, \leq)$ if ρ satisfies the following properties:

1) $\rho(x,y) = \rho(y,x)$ for every $x \in X$ and every $y \in X$;

2) $\rho(x,y) = 0$ if $x = y$;

3) if $x \leq z \leq y$ then $\rho(x,y) \geq \rho(x,z) \vee \rho(z,y)$.

$(X, \leq)$ with ρ is called the quasi-metric space and we write it as $(X, \leq, \rho)$ (or (X, ρ), shortly).

Definition 2 Suppose X be a poset with null element and unit element. Define a mapping ρ:

$$\rho: X \times X \to [0, 1].$$

Mapping ρ is called a normal quasi-metric function if ρ satisfies:

1) $\rho(x,y) = \rho(y,x)$ for any $(x,y) \in X \times X$;
2) $\rho(\phi, u) = 1$; $\rho(x,y) = 0$ if $x = y$;

3) if $x \leq z \leq y$ then $\rho(x,y) \geq \rho(x,z) \vee \rho(z,y)$.

(where ϕ and u are the null element and unit element of $(X, \leq)$, respectively)

$(X, \leq, \rho)$ is called a normal quasi-metric space .

Proposition1 Let $(X, \leq, \rho)$ be a quasi-metric space and M a subset of X . Then $(M, \leq, \rho)$ is also a quasi-metric space .

Definition 3 Suppose $(M_i, \leq_i, \rho_i)$ $(i = 1, 2)$ be two quasi-metric spaces. A mapping ψ:

$$\psi : M_1 \to M_2$$

is called metric-preserving and isomorphic ifthe following hold:

(1) ψ is a bijection;

(2) (order-preserving) $x \leq_1 y$ iff $\psi(x) \leq_2 \psi(y)$;

(3) (metric-preserving) $\rho_1(x,y) = \rho_2(\psi(x), \psi(y))$;

for any two points x and y of M_1.

If there exists a metric-preserving and isomorphic mapping from $(M_1, \leq_1, \rho_1)$ to $(M_2, \leq_2, \rho_2)$ then we say that $(M_1, \leq_1, \rho_1)$ is isomorphic to $(M_2, \leq_2, \rho_2)$.

Remark. The publishment of the concept of metric-preserving and isomorphic enable us to regard the quasi-metric spaces which are isomorphic as equivalent and deal with one of them whose elements are simplier so as to simply the problem.

Proposition 2 If Ψ is a metric-preserving and isomorphic mapping from normal quasi-metric space $(M_1, \leqslant_1, \rho_1)$ to normal quasi-metric space $(M_2, \leqslant_2, \rho_2)$, then $\phi_2 = \Psi(\phi_1)$, $U_2 = \Psi(U_1)$. where ϕ_i, U_i $(i=1,2)$ are the null element and unit element of $(M_i, \leqslant_i, \rho_i)$, respectively.

Definition4 Suppose $(M, \leqslant, \rho)$ be a quasi-metric space, $\{x_n\}$ be a sequence in $(M, \leqslant, \rho)$ and $x \in M$. If $\lim_{n\to\infty} \rho(x_n, x)=0$, we say that point sequence $\{x_n\}$ converges to x, with respect to $\rho(\cdot, \cdot)$. we call x the limit of $\{x_n\}$.

Proposition 3 If $\lim_{n\to\infty} x_n = y$ and $x_n \leqslant x \leqslant y$, (or $x_n \geqslant x \geqslant y$), then $\lim_{n\to\infty} x_n = x$ and $\rho(x,y)=0$.

Proof Since $x_n \leqslant x \leqslant y$, we have $\rho(x_n, y) \geqslant \rho(x,y) \vee \rho(x_n, x)$. By $\lim_{n\to\infty} \rho(x_n, y) = 0$ and the nonnegativity of $\rho(\cdot, \cdot)$, $\rho(x,y)=0$ and $\lim_{n\to\infty} \rho(x_n, x)=0$. This complete the proof.

Definition 5 Suppose $(M, \leqslant, \rho)$ be a quasi-metric space and $x_0 \in M$. Given $r > 0$, the set $\{x \mid x \in M, \rho(x, x_0) < r\}$ is called the open ball with center at x and radius r. We denote it by $O(x_0, r)$, which is also called the r-neighborhood of x_0.

Proposition 4 $x_n \to x_0$ iff for every $\varepsilon > 0$, there exist a integer N such that $n \geqslant N$ implies $x_n \in O(x_0, \varepsilon)$.

Definition 6 Suppose S be a subset of **M.** The set S is called bounded if there exist a open ball $O(x_0, r)$ such that $O(x_0, r) \supseteq S$.

Proposition 5 Every point set of a normal quasi-metric space is bounded.

Definition 7 Suppose $(M, \leqslant, \rho)$ be a quasi-metric space, $S \subset M$. Given $x_0 \in M$. If every neighborhood of x_0 contains infinitely many points of S, then we call x_0 a limit point of S.

Proposition 6 Let $\{x_n\} \subset S$, $x_n \neq x_m (n=m)$ and $\lim_{n\to\infty} x_n = x_0$.

Then x_0 is a limit point of S.

Immediately we can introduce the concepts of open set, closed set, compact set etc. and conclude the corresponding propositions. We omit the discussion.

Specially, given a lattice L with null element and unit element. We define $x \leqslant y \Longleftrightarrow x \vee y = y$, then $(L, \leqslant)$ is a poset with null element and unit element. Thus we can publish the normal quasi-metric space on it. Of course we can publish suitable quasi-metricspace on soft algebra.

§2 Some quasi-metric functions

Example 1 Let M be a subset of euclidean space R^n, $M=\{x \mid x=(x_1, \cdots, x_n),\ 0 \leqslant x_i \leqslant 1,\ i=1,\cdots,n\}$.

For $x=(x_1, \cdots, x_n)$ and $y=(y_1, \cdots, y_n)$ in M, define: $x \leqslant y$ iff $x_i \leqslant y_i$ $(i=1,\cdots,n)$, then $(M, \leqslant)$ is a poset. Define $\rho(x,y)=$

$$= \frac{1}{n^\beta}\left(\sum_{i=1}^{n} |x_i - y_i|^\alpha\right)^\beta (\forall \alpha \geqslant 1, \beta > 0).$$

It is easy to show $(M, \leqslant, \rho)$ is a normal quasi-metric space. Moreover, $x_n \to x$ iff x_n converges to x according to the euclidean distance. Therefore the limit of convergent point sequence in $(M, \leqslant, \rho)$ is unique.

Note. The ρ is not sure the metric on M. For example, take $\alpha=2$ and $\beta=1$, $x=(1,1,\cdots,1)$, $y=(0,0,\cdots,0)$, $z=(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$, then $\rho(x,y)=1$, $\rho(x,z)=\rho(y,z)=\frac{1}{4}$. However $\rho(x,y)=1 > \frac{1}{2}=\rho(x,z)+\rho(y,z)$. Thus (M, ρ) is not a metric space.

Example 2 R^1, according to the partially ordering in Example1, is a poset. For $x \in R^1$ and $y \in R^1$, define

$$\rho(x,y)=\frac{|x-y|}{1+|x-y|}.$$

Since $\frac{x}{1+x}$ is monotone increasing in $(0,\infty)$, it is easy to prove that $(R^1, \leqslant, \rho)$ is a normal quasi-metric space.

<u>Example 3</u> Let $E \in \mathfrak{B}$ ($\mathfrak{B}$ is the family of Borel sets in real line) and $m(E) > 0$ (m is the Lebesgue measure). Take the subset L^0 of $L(E, \mathfrak{B}, m)$, $L^0 \triangleq \{f \mid f \in L,\ 0 \leq f \leq 1\}$. We define a partially ordering "$\leq$" on L^0:

$$f_1(x) \leqslant f_2(x) \quad (\forall x \in X) \Leftrightarrow f_1 \leqslant f_2.$$

In the following we define several quasi-metrics on L, which are all valuable.

1) For $f \in L^0$ and $g \in L^0$, define

$$\rho(f,g)=\begin{cases} \dfrac{\int_E |f-g|\,dm}{\int_E (f+g)\,dm} & \text{if } \int_E (f+g)\,dm \neq 0 \\ 0 & \text{if } \int_E (f+g)\,dm = 0. \end{cases}$$

It is easy to show that $\rho(\cdot,\cdot)$ is a quasi-metric on $(L^0, \leqslant)$. We only show that the condition 3) holds.

If $f \leqslant g \leqslant \varphi$, then
$$\rho(f,\varphi)=\frac{\int_E |f-\varphi|\,dm}{\int_E (f+\varphi)\,dm}=\frac{\int_E (\varphi-f)\,dm}{\int_E (\varphi+f)\,dm}=$$
$$\frac{\int_E (\varphi-g+g-f)\,dm}{\int_E (\varphi+f)\,dm} \geqslant \frac{\int_E (\varphi-g)\,dm}{\int_E (\varphi+g)\,dm}=\rho(g,\varphi);$$
$$\rho(f,\varphi)=\frac{\int_E [(\varphi-g)+(g-f)]\,dm}{\int_E [(\varphi-g)+(g+f)]\,dm} \geqslant \frac{\int_E (g-f)\,dm}{\int_E (g+f)\,dm}=\rho(f,g).$$

(Where we have used the proposition: if $0 < x < y$ then $\dfrac{a+x}{a+y} \geqslant \dfrac{x}{y}$ for any $a > 0$; if $x \geqslant y > 0$, then $\dfrac{a+x}{a+y} \leqslant \dfrac{x}{y}$ for any $a > 0$.)

But $\rho(\cdot,\cdot)$ is not a metric on L^0. In fact, take $E_1 \in \mathfrak{B}$ such that $E_1 \subset E$ and $m(E_1)=\frac{1}{2}m(E)$, $f=\chi_{E_1}$, $g=\chi_{E_1^c}$ and $\varphi \equiv 1$. It is easy to obtain $\rho(f,g)=1$, $\rho(f,\varphi)=\frac{1}{3}$ and $\rho(g,\varphi)=\frac{1}{3}$. Thus $\rho(f,\varphi)+\rho(g,\varphi)=\frac{2}{3} < 1=\rho(f,g)$ and hence $\rho(\cdot,\cdot)$ is not a metric on L^0.

Note. If $m(E) < \infty$, then the convergence on quasi-metric space (L^0, ρ) is equivalent to the convergence on common metric space (L^0, d) (where $d(f,g) = \int_E |f-g| dm$). Moreover, we have

Proposition 7 Let $m(E) < \infty$. In the sence of ommitting the equality almost everywhere, the limit of convergent point sequence of (L^0, ρ) is unique and $\rho(f,g)$ is continuous with respect to f and g.

2. Define
$$\rho(f,g) = \begin{cases} \dfrac{\int_E |f-g| dm}{\int_E (f \vee g) dm} & \text{if} \quad \int_E f \vee g \, dm \neq 0 \\ 0 & \text{if} \quad \int_E f \vee g \, dm = 0 \end{cases}.$$

Then (L^0, ρ) is a normal quasi-metric space.

If $m(E) < \infty$, then we have $\rho(f_n, f) \to 0$ (as $n \to \infty$) iff

$\int_E |f_n - f| dm \to 0$ (as $n \to \infty$) since $0 \leq \int_E (f \vee g) dm \leq \mu(E) < \infty$,

§3 Pseudo-metric space

Definition 8[5] Suppose $a \in (0, \infty)$. A real function
$$\theta : [0,a] \to [0, \infty)$$
is called a T-function if θ is continuous, Strictly monotone and $\theta(0)=0$, $\theta^{-1}(\{\infty\}) \subset \{\infty\}$.

Definition 9 Suppose X be the universe of discourse. A nonnegative and symmetric function δ on $X \times X$ is called a pseudo-metric funtion if there exists a T-function θ such that $\theta \circ \delta$ is a metric function on X. X with δ is called a pseudo-metric space and denoted by (X, δ).

Example. Let (X, λ) be a metric space. Then (X, λ^n) $(n > 0)$ is a pseudo-metric space.

Definition 10 A nonnegative and symmetric function δ on $X\times X$ is called λ-subadditive if there exist a $\lambda\in[0,\infty)$ such that $\delta(x,y)\le\delta(x,z)+\delta(y,z)+\lambda\cdot\delta(x,z)\cdot\delta(y,z)$ for any x,y and z in X.

Theorem Let δ be a nonnegative and symmetric function. If δ is λ-subadditive then δ is a pseudo-metric function on X.

Proof It is clear that $\theta(x)=\dfrac{\ln(1+\lambda x)}{\ln(1+\lambda)}$ is a T-function. Put $\rho=\theta\circ\delta$. For any x,y and z in X we have

$$\rho(x,y)=\frac{\ln(1+\lambda\cdot\delta(x,y))}{\ln(1+\lambda)}\le$$

$$\frac{\ln[1+\lambda[\delta(x,z)+\delta(y,z)+\lambda\cdot\delta(x,z)\cdot\delta(y,z)]]}{\ln(1+\lambda)}=$$

$$\frac{\ln\{1+\lambda\cdot\frac{1}{\lambda}[(1+\lambda\cdot\delta(x,z))(1+\lambda\cdot\delta(y,z))-1]\}}{\ln(1+\lambda)}$$

$$=\frac{\ln(1+\lambda\cdot\delta(x,z))+\ln(1+\lambda\cdot\delta(y,z))}{\ln(1+\lambda)}$$

$$=\rho(x,z)+\rho(y,z).$$

Since δ is nonnegative and symmetric, we also have ρ is nonnegative and symmeytic by $\rho=\theta\circ\delta$. Thus ρ is a common metric function on X. It follow that δ is a pseudo-metric function.

Similarly, we can also introduce some other metric structure and discuss the corresponding proposition. We omit the discussion.

§4 F-metric space

Let X be the universe of discourse and $\mathcal{F}(x)$ the family of fuzzy sets in X. Define the partially ordering "$\leqslant$" : For $\underset{\sim}{A} \in \mathscr{F}(X)$ and $\underset{\sim}{B} \in \mathscr{F}(X)$, $\underset{\sim}{A} \leqslant \underset{\sim}{B} \Leftrightarrow \mu_{\underset{\sim}{A}}(x) \leqslant \mu_{\underset{\sim}{B}}(x)$ ($\forall x \in X$), then $(\mathscr{F}(X), \leqslant)$ is a poset (Of course, the other partially ordering can be also defined on $\mathscr{F}(x)$). Φ and X are the null element and unit element, respectively.

<u>Definition 11</u> A normal quasi-metric ρ is called a F-metric on $(\mathscr{F}(x), \leqslant)$. $(\mathscr{F}(x), \leqslant, \rho)$ is called a F-metric space and we write $(\mathscr{F}(x), \rho)$ briefly.

Example 4 Let $X=\{x_1, x_2, \cdots, x_n\}$. Define a partially ordering "$\leqslant$" : $\underset{\sim}{A} \leqslant \underset{\sim}{B} \Leftrightarrow \mu_{\underset{\sim}{A}}(x_i) \leqslant \mu_{\underset{\sim}{B}}(x_i)$ $(\forall x_i \in X)$.

1) Define
$$\rho(\underset{\sim}{A},\underset{\sim}{B})=\begin{cases} \dfrac{\sum_{i=1}^{n} |\mu_{\underset{\sim}{A}}(x_i)-\mu_{\underset{\sim}{B}}(x_i)|}{\sum_{i=1}^{n} \mu_{\underset{\sim}{A}}(x_i) \vee \mu_{\underset{\sim}{B}}(x_i)} & \left(\text{if } \sum_{i=1}^{n} \mu_{\underset{\sim}{A}}(x_i) \vee \mu_{\underset{\sim}{B}}(x_i) \neq 0\right) \\ 0 & \text{otherwise} \end{cases}$$

It is easy to show that $\rho(\cdot,\cdot)$ is a F-metric on $(\mathscr{F}(x), \leqslant)$.

2) Define
$$\rho(\underset{\sim}{A},\underset{\sim}{B})=\begin{cases} 1 & \underset{\sim}{A} < \underset{\sim}{B} \text{ or } \underset{\sim}{B} < \underset{\sim}{A} \\ 0 & \underset{\sim}{A} \nless \underset{\sim}{B} \text{ and } \underset{\sim}{B} \nless \underset{\sim}{A} \end{cases}.$$

Then ρ is a F-metric on $(\mathscr{F}(x), \leqslant)$. We call this F-metric space a trivial F-metric space.

Note. In this example $\underset{\sim}{A} < \underset{\sim}{B} \Leftrightarrow \underset{\sim}{A} \leqslant \underset{\sim}{B}$ and there exists a x_0 such that $\mu_{\underset{\sim}{A}}(x_0) < \mu_{\underset{\sim}{B}}(x_0)$.

The discussion in the preceding sections can be completely used here.

§5 On the fuzzy near degree

Let $(\mathscr{F}(x), \rho)$ be a F-metric space. Put $n(\underset{\sim}{A},\underset{\sim}{B})=1-\rho(\underset{\sim}{A},\underset{\sim}{B})$, then $n(\cdot,\cdot)$ is a fuzzy near degree on $\mathscr{F}(X)$.

In fact, it is clear that $n(\underset{\sim}{A},\underset{\sim}{B}) \geqslant 0$; $n(\underset{\sim}{A},\underset{\sim}{B})=n(\underset{\sim}{B},\underset{\sim}{A})$; if $\underset{\sim}{A} \geqslant \underset{\sim}{B} \geqslant \underset{\sim}{C}$, then $n(\underset{\sim}{A},\underset{\sim}{C})=1-\rho(\underset{\sim}{A},\underset{\sim}{C}) \leqslant 1-\rho(\underset{\sim}{A},\underset{\sim}{B}) \vee \rho(\underset{\sim}{B},\underset{\sim}{C})=(1-\rho(\underset{\sim}{A},\underset{\sim}{B})) \wedge (1-\rho(\underset{\sim}{B},\underset{\sim}{C}))=n(\underset{\sim}{A},\underset{\sim}{B}) \wedge n(\underset{\sim}{B},\underset{\sim}{C})$. Hence $n(\cdot,\cdot)$ is a fuzzy near degree on $\mathcal{F}(X)$.

Thus we can investigate the behavior of fuzzy near degree by the property of quasi-metric space.

§6 Convex fuzzy set and fuzzy number

We will complete this paper with a discussion of convex fuzzy set and fuzzy number, which is useful in fuzzy programming.

Consider $X \subset R^1$ and X is a L-measurable set, $m(X) < \infty$. We write the class of convex fuzzy sets in X as $\mathcal{F}_\tau(X)$.

Proposition 8 If $\underset{\sim}{A} \in \mathcal{F}_\tau(X)$, then $\mu_{\underset{\sim}{A}}(x) \in L(X)$.

Particularly, we denote the family of fuzzy numbers in X by $\underset{\sim}{X}(R)$, by proposition 8 $\underset{\sim}{A} \in \underset{\sim}{X}(R) \Rightarrow \mu_{\underset{\sim}{A}}(x) \in L(X)$.

Definition 12 Suppose ρ be a quasi-metric on X(R) and $\underset{\sim}{A} \in \underset{\sim}{X}(R)$. $O(\underset{\sim}{A}, \varepsilon)$ is called the ε-neighborhood of A.

Definition 13 Suppose $\underset{\sim}{A}_n \in \underset{\sim}{X}(R)$ $(n=1,2,\cdots)$ and $\underset{\sim}{A} \in \underset{\sim}{X}(R)$. We say that $\{A_n\}$ converges to $\underset{\sim}{A}$ if $\rho(\underset{\sim}{A}_n, \underset{\sim}{A}) \to 0$ $(n \to \infty)$ and write $\underset{\sim}{A}_n \to \underset{\sim}{A}$ $(n \to \infty)$.

It is clear that $\underset{\sim}{A}_n \to \underset{\sim}{A}$ $(n \to \infty) \Leftrightarrow$ for any $\varepsilon > 0$ there exists a integer $N(\varepsilon)$ such that $n \geqslant N(\varepsilon)$ implies $\underset{\sim}{A}_n \in O(\underset{\sim}{A}, \varepsilon)$.

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