A NOTE ON HOMOMORPHISMS ON FUZZY RIGHT-TOPOLOGICAL SEMIGROUPS

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1. INTRODUCTION

Rosenfeld in [2] introduced the concepts of fuzzy groups and fuzzy semigroups. Foster in [1] then formulated the elements of a theory of fuzzy topological groups. In this paper, we extend some of Foster's results on homomorphic images and inverse images to fuzzy right-topological semigroups.

2. PRELIMINARIES

Definition 2.1. A fuzzy topology on a set X is a family $\mathcal{J}$ of fuzzy sets in X such that

(i) $\forall c \in I$, $k_c \in \mathcal{J}$ where k_c have constant membership functions.

(ii) $A, B \in \mathcal{J} \Rightarrow A \cap B \in \mathcal{J}$

(iii) $A_j \in \mathcal{J} \ \forall_j \in J \Rightarrow \bigcup_{j \in J} A_j \in \mathcal{J}$.

The pair $(X, \mathcal{J})$ is called a fuzzy topological space and members of $\mathcal{J}$ are the open fuzzy sets.

Definition 2.2. Let A be a fuzzy set in X and $\mathcal{J}$ a fuzzy topology on X . Then the induced fuzzy topology on A is the family of fuzzy subsets of A which are the intersections with A of $\mathcal{J}$-open fuzzy sets in X . Denote by $\mathcal{J}_A$ the induced fuzzy topology, and by the pair $(A, \mathcal{J}_A)$ the fuzzy subspace of $(X, \mathcal{J})$.

Definition 2.3. Let $(X, \mathcal{J})$, $(Y, \mathcal{U})$ be two fuzzy topological spaces. A mapping f of $(X, \mathcal{J})$ into $(Y, \mathcal{U})$ is fuzzy continuous if for each open fuzzy set V in $\mathcal{U}$ the inverse image $f^{-1}[V]$ is in $\mathcal{J}$.

Definition 2.4. Let $(A, \mathcal{J}_A)$, $(B, \mathcal{U}_B)$ be fuzzy subspaces of fuzzy topological spaces $(X, \mathcal{J})$ and $(Y, \mathcal{U})$ respectively, and let f be a mapping: $(X, \mathcal{J}) \to (Y, \mathcal{U})$. Then f is a mapping of $(A, \mathcal{J}_A)$ into $(B, \mathcal{U}_B)$ if $f[A] \subset B$. Furthermore, f is relatively fuzzy continuous if for each open fuzzy set V' in $\mathcal{U}_B$, the intersection $f^{-1}[V'] \cap A$ is in $\mathcal{J}_A$.

Foster in [1] has formulated results for fuzzy topological groups. We extend these results to the more general case:

Definition 2.5. Let X be a semigroup and $\mathcal{J}$ a fuzzy topology on X . Let G be a fuzzy semigroup in X with induced topology $\mathcal{J}_G$. Then G is a fuzzy right-topological semigroup in X iff for each $s \in X$, the mapping $f_s : x \to xs$ of $(G, \mathcal{J}_G) \to (G, \mathcal{J}_G)$ is relatively fuzzy continuous.

3. THE MAIN RESULTS

Theorem 3.1. Given semigroups X, Y and a homomorphism $f: X \to Y$, let $\mathcal{U}$ be the fuzzy topology on Y and $\mathcal{J}$ be the fuzzy topology on X such that $\mathcal{J} = f^{-1}[\mathcal{U}]$. Let G be a fuzzy right-topological semigroup in Y . Then $f^{-1}[G]$ is a fuzzy right-topological semigroup in X.

Proof. We must show that the mapping $\rho_s : x \to xs$ of $f^{-1}[G] \to f^{-1}[G]$ is relatively fuzzy continuous. Let U' be an open fuzzy set in $\mathcal{J}_{f^{-1}[G]}$ on $f^{-1}[G]$. Note that since f is a fuzzy continuous mapping from: $(X, \mathcal{J}) \to (Y, \mathcal{U})$, it is a relatively fuzzy continuous map of $(f^{-1}[G], \mathcal{J}_{f^{-1}[G]})$ into $(G, \mathcal{U}_G)$. Note also that there exists an open fuzzy set $V' \in \mathcal{U}_G$ such that $f^{-1}[V'] = U'$.

The membership function of $\rho_s^{-1}[U']$ is given by

$$\begin{aligned}
\mu_{\rho_s^{-1}[U']}(x) &= \mu_{U'}[\rho_s(x)] \\
&= \mu_{U'}(xs) \\
&= \mu_{f^{-1}[V']}(xs) \\
&= \mu_{V'}(f(xs)) \\
&= \mu_{V'}\ f(x)f(s) .
\end{aligned}$$

But $\mu_{\rho_t^{-1}[V']}(f(x)) = \mu_{V'}[\rho_{f(s)}(f(x))]$

$$= \mu_{\rho_{f(s)}^{-1}[V']}(f(x))$$

$$= \mu_{f^{-1}[\rho_{f(s)}^{-1}[V']]}(x),\ t \in Y .$$

By hypothesis the mapping $\rho_t : y \to yt$ of $(G, \mathcal{U}_G) \to (G, \mathcal{U}_G)$ is relatively fuzzy continuous. Hence, $\rho_t^{-1}[U'] \cap f^{-1}[G] = f^{-1}[\rho_{f(s)}^{-1}[V']] \cap f^{-1}[G]$ is open in the induced fuzzy topology on $f^{-1}[G]$.

Theorem 3.2. Given semigroups X, Y and a homomorphism f of X onto Y , let $\mathcal{J}$ be the fuzzy topology on X and $\mathcal{U}$ be the fuzzy topology on Y such that $f(\mathcal{J}) = \mathcal{U}$. Let G be a fuzzy right-topological semigroup in X . If the membership function μ_G of G is f-invariant, then f[G] is a fuzzy right-topological semigroup in Y .

Proof. From Rosenfeld [4], f[G] is a fuzzy semigroup. We must show the mapping $\rho_t : y \to yt$ from $(f[G], \mathcal{U}_{f[G]}) \to (f[G], \mathcal{U}_{f[G]})$ is relatively fuzzy continuous.

Note that f is relatively fuzzy open; for if $U' \in \mathcal{J}_G$, there exists $U \in \mathcal{J}$ such that $U' = U \cap G$ and by the f-invariance of μ_G ,

$$f[U'] = f[U] \cap f[G] \in \mathcal{U}_{f[G]} .$$

Let V' be an open fuzzy set in $\mathcal{U}_{f[G]}$. Since f is onto, for each t ε Y there exists s ε X such that t = f(s) . Hence,

$$\begin{aligned}
\mu_{f^{-1}[\rho_t^{-1}[V']]}(x) &= \mu_{f^{-1}[\rho_{f(s)}^{-1}[V']]}(x) \\
&= \mu_{\rho_{f(s)}^{-1}[V']}(f(x)) \\
&= \mu_{[V']}[f(x)f(s)] \\
&= \mu_{[V']}f(xs) \\
&= \mu_{f^{-1}[V']}(\rho_s(x)) \\
&= \mu_{(\rho_s^{-1} \circ f^{-1})[V']}(x) .
\end{aligned}$$

By hypothesis, $\rho_s : x \to xs$ is a relatively fuzzy continuous map: $(G, \mathcal{J}_G) \to (G, \mathcal{J}_G)$ and f is a relatively fuzzy continuous map: $(G, \mathcal{J}_G) \to (f[G], \mathcal{U}_{f[G]})$. Hence, $f^{-1}[\rho_t^{-1}[V']) \cap f[G]] = f^{-1}[\rho_t^{-1}[V']] \cap G$ is open in $\mathcal{U}_G$. Since f is relatively fuzzy open, $(f \circ (f^{-1}))[\rho_t^{-1}[V']] \cap f[G] = \rho_t^{-1}[V'] \cap f[G]$ is open in $\mathcal{U}_{f[G]}$.

REFERENCES

1. D. H. Foster, Fuzzy Topological Groups, Journal of Mathematical Analysis and Applications, 67(1979), 549-564.

2. A. Rosenfeld, Fuzzy Groups, Journal of Mathematical Analysis and Applications, 35(1971), 512-517.