SOLVING THE MATHEMATICAL PROGRAMMING PROBLEM WITH A NEW FORMULATION OF FUZZY OBJECTIVE.

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INTRODUCTION.- We develop the fuzzy objective concept that was introduced in [9]. The practical situation which gives rise to this new concept is as follows. Suppose us a decisor which for each of the products he sells, has a value interval to fix the public sale price and, therefore, to assign the profit. Together, with the price fixed for each item, he has established a function which represents the accomplishment degree's of his decision, this degree depending on such factors as the season of the year, quality of the competition in each product, etc. These factors make him not mark the items with the maximum profit since, in such a case, the competition, for instance, could profit of it by mean of reducing the prices for the same articles. Therefore, the decisor has, in order to optimize his profir, an objective which in a wide sense, is a fuzzy one.

As it is well know, a Linear Programming problem is stated in the following terms,

$$\underset{x \in X}{\text{Max:}}\ cx \qquad (1)$$

and can be represented by means of the triplet (R^n, X,c). Such a representation has to be understood in the sense of finding an element $x^* \in X$ R^n such that,

$$\forall x \in R^n \rightarrow cx^* \geqslant cx$$

The possible fuzzifications which could be determined on (1), were studied in [9]. Three kinds of Fuzzy Linear Programming (FLP) problems appear according with the fuzziness let be present either in the objective, or in the fuzzy constraint set or both objective and constraint set.

The main aim of this paper is concerning with FLP problems - that the fuzziness is relative to the objective only. For this problem we investigate a fuzzy solution, as corresponding to the inaccurate statement of the problem, and not a crisp solution. In - - this way, we must first determine the concept of fuzzy objective - which we shall use..

FUZZY OBJECTIVE: A NEW FORMULATION.- Suppose an FLP problem with - fuzzy objective,

$$\begin{aligned} &\underset{\sim}{\text{Max}}: \; cx \\ &\text{s.t:} \\ &\qquad Ax \leqslant b \\ &\qquad x \geqslant 0 \end{aligned} \tag{2}$$

in which the symbol "$\sim$" implies precisely that the objective is a fuzzy one in the sense of the following,

DEFINITION.- A Fuzzy Objective is a fuzzy set of the objective set

This intuitive definition agrees with the Fuzzification Principle of Goguen [4] and implies that, given a Fuzzy Mathematical - Programming problem (either linear or not) as,

$$\underset{\substack{\sim \\ x\varepsilon X}}{\text{Max}}: \; f(x) \tag{3}$$

with,

$$f: R^n \longrightarrow R$$

the fuzzy optimization shown by the symbol "$\sim$" implies the existence of a membership function

$$\phi: \mathfrak{F}(R^n) \longrightarrow [0,1]$$

where,

$$\mathfrak{F}(R^n) = \{f \,/\, f: R^n \longrightarrow R\}$$

is the objective set to which the above definition refers. Therefore generally speaking, a fuzzy objective is a fuzzy set $\phi \varepsilon F[\mathfrak{F}(R^n)]$.

Particulary, in the case of considering only linear objective functions, it is known that,

$$\mathfrak{F}(R^n) \equiv R^n$$

and thus, for FLP problems, a fuzzy objective is a fuzzy set

$$\phi: R^n \longrightarrow [0,1]$$

On other hand, as it occurs in unfuzzy mathematical programming - problems with the objective function, a fuzzy objective induces, among the admissible alternatives, a fuzzy preorder. This was shown in [9].

The situation we formulate in the introduction is a little different from the above stated fuzzy objective formulation. In fact, if we limit ourselves exclusively to the objective, without considering the constraint set, we formulate now the existence of a membership function,

$$\phi_i : R \longrightarrow [0,1], \quad i = 1,2,\ldots,n$$

related to each cost (profit) taking part in the objective, i.e., - we are considering with the fuzzy optimization of (3) an n-vector - function of membership functions $(\phi_1,\ldots,\phi_n)$.

No doubt, this situation, both in the linear case and in the - non-linear with suitable hipotheses, implicitely assumes the definition of a fuzzy objective. This would be defined as,

$$\forall c \in R^n, \quad \phi(c) = \operatorname*{Inf}_i \phi_i(c_i), \quad c = (c_1,\ldots,c_n) \qquad (4)$$

in the linear case, and as

$$\forall f \in \mathfrak{F}(R^n), \quad \phi(f) = \operatorname*{Inf}_i \phi_i(f_i), \quad f = (f_1,\ldots,f_n)$$

in the non-linear case for a vector objective function $f=(f_1,..f_n)$

From now on, we will always use the vector form whenever we - talk about fuzzy objective which will formally take the form (5).

Let be a FLP problem,

$$\begin{array}{l} \underset{\sim}{\text{Max}}:\ cx \\ \text{s.t:} \\ \qquad Ax \leqslant b \\ \qquad x \geqslant 0 \end{array} \qquad (5)$$

with,

$$\phi = (\phi_1,\ldots,\phi_n); \quad \phi_i : R \rightarrow [0,1], \quad i = 1,2,\ldots,n$$

Let the membership function of the fuzzy objective, according to (4), be

$$\forall c \in R^n, \quad \psi(c) = \operatorname*{Inf}_i \phi_i(c_i)$$

In [9] it was shown that the fuzzy solution of (5) could be found-with the help of the following problem,

$$\begin{array}{l} \text{Max: } cx \\ \text{s.t:} \\ \qquad \psi(c) \geqslant 1-\alpha \\ \qquad Ax \leqslant b \\ \alpha\varepsilon[0,1], x \geqslant 0,\ c\varepsilon R^n \end{array} \tag{6}$$

For the definite solution to (5) we try the following,

RESULT.- Suppose the FLP problem (5). If membership functions,

$$\phi_i : R \longrightarrow [0,1]\ ,\ i = 1,2,\ldots,n$$

are continuous and strictly monotones, the fuzzy solution of (5) - is given by the optimal solution of the parametric linear problem,

$$\begin{array}{l} \text{Max: } z(\beta) = \eta(\beta)x \\ \text{s.t:} \\ \qquad Ax \leqslant b \\ \qquad x \geqslant 0,\ \alpha\varepsilon[0,1] \end{array} \tag{7}$$

where: $\eta(\beta) = [\eta_1(\beta),\ldots,\eta_n(\beta)]$ is a n-vector function with,

$$\eta_i : [0,1] \longrightarrow R$$

Proof: According to (6), the following will have to be solved,

$$\begin{array}{l} \text{Max: } cx \\ \text{s.t:} \\ \qquad \psi(c) \geqslant 1-\alpha \\ \qquad Ax \leqslant b \\ \qquad x \geqslant 0,\ \alpha\varepsilon[0,1] \end{array}$$

being,

$$\forall c\varepsilon R^n,\ \psi(c) = \operatorname*{Inf}_i \phi_i(c_i)$$

But if,

$$\psi(c) \geqslant 1-\alpha \rightarrow \operatorname*{Inf}_i \phi_i(c_i) \geqslant 1-\alpha \rightarrow$$

$$\phi_i(c_i) \geqslant 1-\alpha,\ i = 1,2,\ldots,n$$

As ϕ_i is continuous and strictly monotone,

$$\phi_i(c_i) \geqslant 1-\alpha \ \rightarrow\ c_i \geqslant \phi_i^{-1}(1-\alpha)$$

and therefore, we have,

$$\begin{aligned}&\text{Max: } \sum_i c_i x_i\\ &\text{s.t:}\\ &\qquad c_i \geqslant \phi_i^{-1}(1-\alpha)\ ,\ i = 1,2,\ldots,n\\ &\qquad Ax \leqslant b\\ &\qquad x \geqslant 0,\ \alpha\varepsilon[0,1]\end{aligned} \tag{8}$$

But this problem is equivalent to,

$$\begin{aligned}&\text{Max: } \sum_i c_i x_i.\\ &\text{s.t:}\\ &\qquad c_i = \phi_i^{-1}(1-\alpha)\ ,\ i = 1,2,\ldots,n\\ &\qquad Ax \leqslant b\\ &\qquad x \geqslant 0,\ \alpha\varepsilon[0,1]\end{aligned} \tag{9}$$

that is, every optimal solution of (9) is also optimal for (8).

Thus, we still have to solve,

$$\begin{aligned}&\text{Max: } z(\alpha) = \sum_i \phi_i^{-1}(1-\alpha)x_i\\ &\text{s.t:}\\ &\qquad Ax \leqslant b\\ &\qquad x \geqslant 0,\ \alpha\varepsilon[0,1]\end{aligned} \tag{10}$$

which coincides with (7) only by taking,

$$\eta_i(\beta) = \phi_i^{-1}(1-\alpha)\ ,\ i = 1,2,\ldots,n \quad \blacksquare$$

Remark: As it is well known, in the optimum $z^*(\alpha)$ is a piecewise - continuous linear and convex function. On other hand, the- membership functions ϕ_i should be taken strictly increasing or decreasing, depending whether the problem is of maximization or minimization.

EXAMPLE.- Suppose a decisor sells two articles x_1 and x_2. The profit of the sale of x_2 is legally fixed but, for x_1, said profit may be freely fixed in the value interval [40,115], his decision depending on the market situation. For his interest, the degree of accomplishment which has the choice of each profit is determined by

$$\phi_1(c_1) = \begin{cases} 0 & \text{if } 40>c_1 \\ (c_1-40)^2/5625 & \text{if } 40\leqslant c_1\leqslant 115 \\ 1 & \text{if } \quad c_1>115 \end{cases} \tag{11}$$

and the constraints to which the quanties of articles to be sold - are submitted are,

$$X = \{x\varepsilon R^2 \ / \ 3x_1 - x_2 \leqslant 2; \ x_1 + 2x_2 \leqslant 3; \ x_1, x_2 \geqslant 0\}$$

The decisor wants to maximize his profit, i.e., to solve the problem,

$$\underset{x\varepsilon X}{\text{Max:}} \ c_1x_1 + 75x_2$$

As in this case only a membership function takes part, it is clear, according to (4) that,

$$\forall c\varepsilon R^2, \ \psi(c) = \phi_1(c_1)$$

and according to (7), or similarly, to (10), the following must be solved,

$$\begin{aligned} &\text{Max: } z(\alpha) = (40+75\sqrt{1-\alpha})x_1 + x_2 \\ &\text{s.t:} \\ &\qquad 3x_1 - x_2 \leqslant 2 \\ &\qquad x_1 + 2x_2 \leqslant 3 \\ &\qquad x_i \geqslant 0, \ \alpha\varepsilon[0,1] \end{aligned}$$

Solving, it is found that the optimal solution of this parametric-linear problem is,

$$x_1 = x_2 = 1 \quad \text{if } \alpha\varepsilon[0,1]$$

thus the fuzzy solution being the fuzzy set,

$$z^*(\alpha) = \{75+c_1; \ \alpha\}$$

with,

$$\alpha = (c_1 - .40)^2/5625$$

Remark: Let us note that the linearity of the FLP problems which have been considered, is never lost by the special form which the membership functions may have, even if these are non-linear (as in the example solved). This fact is only related to the parameter used, therefore the global linearity of the problems not being affected.
On other hand, fuzzy constraints can be considered with the above fuzzy objective approach's. This fact assumes not theoretical complications.

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