Decomposition problem of fuzzy relations

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Abstract.The paper deals with a problem of decomposition of a binary, fuzzy relation defined in cartesian product of a finite space.We shall derive an algorithm leading to its decomposition.It is based on several facts concerned with fuzzy relational equations and within a finite sequence of steps provides the decomposition or indicates the relation is nondecomposable.

Keywords:decomposition of relation,fuzzy relational equation.

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1. Preliminary remarks and problem statement.

In the paper we shall deal with a new and unsolved problem in a class of fuzzy relational equations, which are treated as a quite new tool for analysis, modelling and synthesis problems in nondeterministic systems. Despite many analytical results in the area of solving fuzzy relational equations e.g. [6], [7], [9] and characterization of a family of solutions (cf. [1][3][4][5]) the decomposition problem of a binary fuzzy relation has not been attacked.

The decomposition problem is formulated as follows: for a given relation $R, R=[r_{ij}], i,j=1,2,\dots,n$, defined in the cartesian product $\mathbf{X}\times\mathbf{X}$, $\mathbf{X}=\{x_1,x_2,\dots,x_n\}$, $\mathrm{card}(\mathbf{X})=n$, find a fuzzy relation $M=[m_{ij}], i,j=1,2,\dots,n$ defined in $\mathbf{X}\times\mathbf{X}$ which fulfils condition:

$$M \,\square\, M = R, \tag{1}$$

where "$\square$" stands for a max-t composition operator [6], therefore (1) is read in the following manner:

$$\bigvee_{k=1}^{n} \left(M(x_i,x_k) \ t\ M(x_k,x_j)\right) = R(x_i,x_j)$$

$i,j=1,2,\dots,n$. "t" denotes triangular norm (cf. [6]) and $\vee$ stands for maximum; $\bigvee_{i=1}^{n} b_i = \max_{1\le i\le n} b_i$. Moreover we shall adopt usually applied notation as given in [2]: $F(\mathbf{X}\times\mathbf{X})$ denotes a family of all fuzzy relations (matrices) defined on $\mathbf{X}\times\mathbf{X}$, while $F(\mathbf{X})$ stands for a family of all fuzzy sets in $\mathbf{X}$. By I_n we denote a set of first "n" natural numbers:

$$I_n=\{1,2,\dots,n\}$$

2. Solution of the problem

Let us introduce a family of fuzzy matrices $\mathcal{M} \in F(\mathbf{X} \times \mathbf{X})$ constitued ny by n^3 matrices $M^{(h,i,k)}$ defined as

Definition 1.

$$M^{(h,i,k)} = [m_{pq}^{(h,i,k)}], p,q \in I_n, h,i,k \in I_n$$

$$m_{pq}^{(h,i,k)} = \begin{cases} r'_{hk}, \text{if } p \in \{h,i\} \text{ and } q \in \{i,k\} \\ 0, \text{otherwise} \end{cases}$$

where r'_{hk} is calculated from the equation $r'_{hk} t r'_{hk} = r_{hk}$. For instance, bearing in mind several examples of triangular norms we get respectively:

- $\min(r'_{hk}, r'_{hk}) = r_{hk}$, $r'_{hk} = r_{hk}$,
- $r'_{hk} \cdot r'_{hk} = r_{hk}$, $r'_{hk} = \sqrt{r_{hk}}$,
- $$\log_w\left(1 + \frac{(w^{r'_{hk}} - 1)(w^{r'_{hk}} - 1)}{(w-1)}\right) = r_{hk},$$

$$r'_{hk} = \log_w[1 + \sqrt{(w-1)(w^{r_{hk}} - 1)}] , \quad 0 < w < \infty, w \neq 1.$$

The following theorem holds:

Theorem 1.

For any $h,i,k \in I_n$ the fuzzy relation $M^{(h,i,k)}$ satisfies formula

$$M^{(h,i,k)} \square M^{(h,i,k)} = R_{h,k} \tag{2}$$

where $R_{h,k}$ is a fuzzy relation with the membership function:

$$R_{h,k}(x_i, x_j) = \begin{cases} r_{hk}, \text{if } i=h \text{ and } j=k \\ 0, \text{otherwise} \end{cases}$$

Proof. Rewritting (2) in terms of membership functions it yields:

$$\bigvee_{j=1}^{n} \left(m_{hj}^{(h,i,k)} \, t \, m_{jk}^{(h,i,k)}\right) = r_{hk}$$

because due to the Definition 1 $m_{hi}^{(h,i,k)}$ t $m_{ik}^{(h,i,k)}$ $=r_{hk}$ only for $i=j$ and $m_{hj}^{(h,i,k)}=m_{jk}^{(h,i,k)}=0$ for $j\in I_n-\{i\}$.

For a given n^2-tuple $(i_1,i_2,\dots,i_m)$of elements of I_n,we fix our attention on n^2 matrices $M^{(h,i_m,k)}\in\mathcal{M}$,$h,k\in I_n$,$m\in I_{n^2}$ such that

$$m=k+(h-1)\,n \tag{3}$$

holds.Observe that assigned a pair $(h,k)\in I_n\times I_n$,there exists an unique $m\in I_{n^2}$ such that (3) holds,and vice versa.

We recall the following lemma[7]:

Lemma 1. If $S\leq T$,$S,T\in F(X\times X)$then $S\square R\leq T\square R$ for any $R\in F(X\times X)$.

Now ,if we put down for any $m\in I_{n^2}$:

$$S^{(m)}=M^{(h,i_m,k)}$$

we get the following:

Theorem 2.

If the fuzzy relation S given by

$$S=\bigvee_{m=1}^{n^2} S^{(m)}$$

verifies an inequality $S\leq S^{-1}\circledast R$ then (1)holds; $\circledast$ is an inverse composition operator applied while solving fuzzy relational equations with sup-t composition (cf. [6],[7]).S^{-1} denotes a transposition of S,$s_{ik}^{-1}=s_{ki}$.

Proof.From Theorem 1,for any given pair$(h,k)\in I_n\times I_n$ we have:

$$\bigvee_{j=1}^{n}(s_{hj}\ t\ s_{jk})\geqslant\bigvee_{j=1}^{n}(s_{hj}^{(m)}ts_{jk}^{(m)})\geqslant r_{hk} \tag{4}$$

with m defined by (3) .The abovestated formula is fulfilled for all indices $h,k\in I_n$,so $S\square S\geqslant R$.Furthermore from [6],[7]we deduce:

$$S\square S\leq(S^{-1}\circledast R)\square S\leq R\ . \tag{5}$$

Therefore from (4) and (5) we obtain the thesis.

The flowchart of the algorithm of decomposition of the fuzzy relation is given in Fig.1.

Fig.1 ⟶

If the algorithm does not provide S,we speak about nondecomposable fuzzy relation R.Then we can look for the relation S which fits "the best" in sense of a performance index imposed .Consider for instance the sum of Euclidean distance equal to

$$Q=\sum_{i=1}^{n}\sum_{j=1}^{n}(s_{ik}ts_{kj}-r_{ij})^2.$$

Then we get the optimization problem:

$$\min Q$$
$$0\leq s_{ij}\leq 1$$
$$i,j=1,2,\ldots,n$$

In order to solve it we use an iteration scheme proposed in [8]:

$$s_{uv}^{(K+1)}=s_{uv}^{(K)}-a_K\,\partial Q/\partial s_{uv}\Big|_{s_{uv}=s_{uv}^{(K)}}$$

$u,v=1,2,\ldots,n,K=0,1,\ldots,$ with an initial condition $s_{uv}^{(0)}=0$, and a_K being a parameter adjusted in an experimental way.

We shall illustrate the application of the method provided with the help of a numerical example.Let a fuzzy relation R is given as follows:

$$R=\begin{bmatrix}0.6 & 0.4 & 0.5\\ 0.4 & 0.6 & 0.5\\ 0.3 & 0.3 & 0.7\end{bmatrix}$$

We specify the triangular norm as a minimum;hence φ stands for a well known α-operator introduced by Sanchez [9]:

$$a\alpha b=\begin{cases}1, & a\le b\\ b, & a>b\end{cases}$$

Following the scheme in Fig.1 we have:

$$S=\begin{bmatrix}0.6 & 0.4 & 0.5\\ 0.4 & 0.6 & 0.5\\ 0.3 & 0.0 & 0.7\end{bmatrix}$$

and $S\Box S=R$.

3. Concluding remarks

We have derived the algorithm of decomposition of the fuzzy relation with the help of several results given in theory of fuzzy relational equations with triangular norms. We would like to underline an applicational importance of the problem under discussion. Let us consider one of the problems in fuzzy relational system where the use of the decomposition algorithm is straightforward. The fuzzy system of the first order is described by means of the equation:

$$X_{k+T}=X_k\Box R$$

$X_k, X_{k+T}\in F(\mathbf{X})$, $R\in F(\mathbf{X}\times\mathbf{X})$, where T denotes a sampling interval between discrete time moments. We are interested in finding the state of the system in the k+T/2 time moment. Thus we are looking for a fuzzy relation $S\in F(\mathbf{X}\times\mathbf{X})$, such that $X_{k+T/2}\Box S=X_k\Box S$ and $X_{k+T}=X_{k+T/2}\Box S$, so

$$X_{k+T}=X_k\Box S\Box S \qquad =X_k\Box R\ .$$

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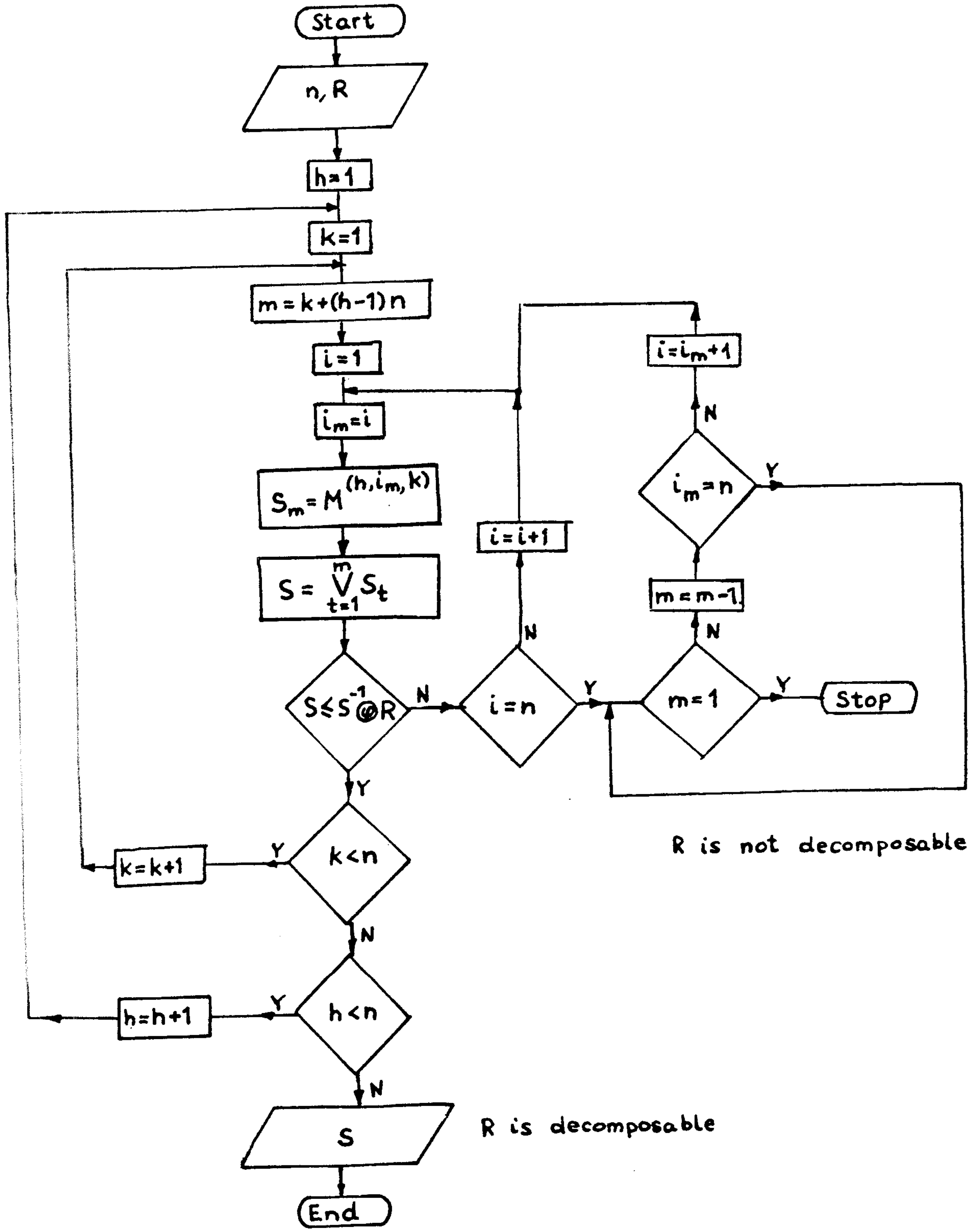
Start
n, R
h=1
k=1
m = k+(h-1)n
i=1
$i_m = i$
$S_m = M^{(h,i_m,k)}$
$S = \bigvee_{t=1}^{m} S_t$
$S \le S^{-1} @ R$
N
Y
i=n
$i = i+1$
m=1
Stop
m = m-1
$i_m = n$
$i = i_m + 1$
R is not decomposable
k<n
k=k+1
h<n
h=h+1
S
R is decomposable
End

Fig.1