System of equations in a linear lattice

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Set of all solutions of max-min system of equations is descripted. Presented algorithm is useful in determination of all solutions.

1. Introduction. Consideration of fuzzy relation equation (cf. [4]) can be restricted to the case similar to linear agebraic equation $A\,x = b$, where addition and multiplication are changed for the lattice operations max and min.

Sanchez [4] gives a characterization of solvability of fuzzy relation equations in infinitely distributive lattice and presentes formulas for the extremal solutions. Familly of all solutions is descripted in a very simpler case of the lattice $L = [0,1]$ and fuzzy relations on a finite set (cf. [2], [3]). Considerations from papers [2] and [3] can be generalized to the case of arbitrary bounded linear lattice. We rewrite some results from these papers and modify algorithm from paper [2].

2. Extremal solutions. Let L be a bounded linear lattice with bounds denoted by 0 and 1, and let m, n be fixed positive integers. We ask for solution $x \in L^m$ of the system of equations $x \circ A = b$, i.e.

$$\max_{i \in I} \min(x_i, a_{ij}) = b_j \quad \text{for} \quad j \in J , \tag{1}$$

where

$$A \in L^{m \times n}, \quad b \in L^n , \; I = \{1, 2, \ldots, m\} , \quad J = \{1, 2, \ldots, n\} .$$

The set of all such solutions we denote by $X(A,b)$. After Sanchez [4] we have

Theorem 1. $X(A,b) \neq \emptyset$ iff $u \circ A = b$, where

$$u_i = \min_{j \in J}(a_{ij} \rightarrow b_j) \quad \text{for} \quad i \in I , \tag{2}$$

$$a \rightarrow b = \begin{cases} 1 , \text{ if } & a \leqslant b \\ b , \text{ if } & a > b \end{cases} \quad \text{for } a, b \in L . \tag{3}$$

Furthermore solution (2) is the greatest one in $X(A,b) \neq \emptyset$.

Here $X(A,b)$ is considered as poset (partially ordered set) with order relation "$\leqslant$" restricted to $X(A,b)$ from the lattice L^m, i.e.

$$(v \leqslant z) \Leftrightarrow (v_i \leqslant z_i \text{ for } i \in I) \quad \text{for } v,\ z \in L^m . \tag{4}$$

In general, $X(A,b)$ does not form a sublattice in L^m and has not the least element (cf. [2]). So we must consider lower bounds of $X(A,b)$ in the form of minimal elements of this set.

Definition 1 (cf. [1], p.4). Let X denote a subset of arbitrary poset. A minimal element of X is an element $z \in X$ such that

$$(x \leqslant z) \Rightarrow (x = z) \quad \text{for} \quad x \in X . \tag{5}$$

Similarly, $z \in X$ is a maximal element of X if

$$(z \leqslant x) \Rightarrow (x = z) \quad \text{for} \quad x \in X . \tag{6}$$

Set of all minimal elements in $X(A,b)$ we denote by $X_0 = X_0(A,b)$.

Example 1. Let $m = 3$, $n = 4$, $L = [0,7]$,

$$A = \begin{vmatrix} 1 & 3 & 4 & 2 \\ 3 & 3 & 7 & 6 \\ 2 & 0 & 4 & 4 \end{vmatrix} , \quad b = [3, 3, 4, 4] .$$

Formulas (2) and (3) gives $u = [7, 4, 7]$ and we get $u \circ A = b$. Hence $X(A,b) \neq \emptyset$ and $X_0 = \{y, z\}$, $y = [0, 4, 0]$, $z = [0, 3, 4]$.

3. Characterization of $X(A,b)$. We assume that $X(A,b) \neq \emptyset$ for given A and b. Because of isotonity of the lattice operations in (1) (cf. [1],p.9) we have (cf. [2]).

Lemma 1. $X(A,b)$ is a convex subset of L^m, i.e. for any $y, z \in X(A,b)$, the lattice interval

$$[y, z] = \{x \in L^m \mid y \leqslant x \leqslant z\} \tag{7}$$

is contained in $X(A,b)$.

In example 1 we have $[y, u] \cup [z,u] \subset X(A,b)$. We prove that any solution of (1) is contained in a certain lattice interval determined by $z \in X_0$ and solution (2) (cf. theorem 2).

Now let us consider $k \in I^n$, $k = [k_1, \ldots, k_n]$ and put

$$I_k = \{k_1, \ldots, k_n\} \subset I, \quad J_k(i) = \{j \in J \mid i = k_j\} \quad \text{for} \quad i \in I , \tag{8}$$

$$k(b) \in L^m , \quad k(b)_i = \begin{cases} \max\limits_{j \in J_k(i)} b_j , & \text{if } i \in I_k \\ 0 , & \text{if } i \notin I_k \end{cases} \quad \text{for } i \in I , \tag{9}$$

where formula (9) is a special case of mapping used for function image of fuzzy set (cf. Zadeh [5]), if we consider $k \in I^n$ and $b \in L^n$ as functions $k : J \to I$, $b : J \to L$ (observe that $J_k(i) = k^{-1}(\{i\})$ for $i \in I$).

Since lattice L is linear, then for any $j \in J$, all elements of the set $\{\min(x_i, a_{ij})\}_{i \in I}$ are comparable and this finite set has the greatest element for certain index $i = k_j$. Thus we have

Lemma 2 (cf. [2]). For any $x \in X(A,b)$ there exists $k \in I^n$, such that

$$\min(x_{k_j}, a_{k_j,j}) = b_j \quad \text{for } j \in J. \tag{10}$$

Set of all k from lemma 2 is denoted by $K(x) \subset I^n$ for $x \in X(A,b)$, i.e.

$$K(x) = \{k \in I^n \mid k \text{ fulfils } (10)\} . \tag{11}$$

Definition 2 (cf. [2], [3]). By a projection of solution $x \in X(A,b)$ with respect to $k \in K(x)$ we name $x^k = k(b) \in L_0^n$, i.e. (cf.(9))

$$x_i^k = \begin{cases} \max\limits_{j \in J_k(i)} b_j , & \text{if } i \in I_k \\ 0 , & \text{if } i \notin I_k \end{cases} \quad \text{for } i \in I , \tag{12}$$

where

$$L_0 = \{0, b_1, \ldots, b_n\} \subset L . \tag{13}$$

Set of all projections of all solutions we denote by $X_1 = X_1(A,b)$.

Example 2. Let us consider solutions u, y, z from example 1. We have $K(u) = \{k^1, \ldots, k^{12}\}$, $K(y) = \{k^9\}$, $K(z) = \{k^{12}\}$, where

$k^1 = [2,1,1,2]$, $k^2 = [2,1,1,3]$, $k^3 = [2,1,2,2]$, $k^4 = [2,1,2,3]$,
$k^5 = [2,1,3,2]$, $k^6 = [2,1,3,3]$, $k^7 = [2,2,1,2]$, $k^8 = [2,2,1,3]$,
$k^9 = [2,2,2,2]$, $k^{10} = [2,2,2,3]$, $k^{11} = [2,2,3,2]$, $k^{12} = [2,2,3,3]$.

Then projections $x^i = x^{k^i}$ for $i = 1,2,\ldots, 12$ have the form

$x^1 = x^7 = [4,4,0]$, $x^2 = x^8 = [4,3,4]$, $x^3 = [3,4,0]$, $x^4 = x^5 = [3,4,4]$,
$x^6 = [3,3,4]$, $x^9 = y = [0,4,0]$, $x^{10} = x^{11} = [0,4,4]$, $x^{12} = z = [0,3,4]$.

Let observe that minimal solutions y and z are special cases of projections, and all projections are solutions of (1) (cf. lemma 1).

Now we reprove (cf. [2])

Lemma 3. Any projection of solution is also a solution and we have

$$x^k \leqslant x\ , \quad k \in K(x^k) \quad \text{for} \quad x \in X(A,b),\ k \in K(x)\ . \tag{14}$$

Proof. Let $x \in X(A,b)$, $k \in K(x)$, $i \in I$. If $i \notin I_k$ then by (12) $x_i^k = 0 \leqslant x_i$. If $i \in I_k$, then $i = k_j$ for $j \in J_k(i)$ and by (10), (12) we get

$$x_i^k = \max_{j \in J_k(i)} b_j = \max_{j \in J_k(k_j)} \min(x_i, a_{ij}) \leqslant x_i\ .$$

Thus $x^k \leqslant x$ and by isotonity of max and min in (1) we get

$$x^k \circ A \leqslant x \circ A = b\ . \tag{15}$$

Now let $s \in J$. Using (1) and (10) we get

$$(x^k \circ A)_s = \max_{i \in I} \min(x_i^k, a_{ij}) \geqslant \min(x^k_{k_s}, a_{k_s,s}) =$$

$$= \min(\max_{j \in J_k(k_s)} b_j, a_{k_s,s}) \geqslant \min(b_s, a_{k_s,s}) = b_s\ ,$$

because $b_s \leqslant a_{k_s,s}$ by (10). Therefore $x^k \circ A \geqslant b$, which together with (15) proves that $x^k \in X(A,b)$. Now above inequality changes for equality and we get

$$\min(x^k_{k_s}, a_{k_s,s}) = b_s \quad \text{for} \quad s \in J\ ,$$

which denote that (cf.(11)) $k \in K(x^k)$. It finishes the proof of lemma 3.

Now we list some useful corollaries.

Corollary 1. As a direct ~~afxiexxx~~ consequence of lemma 3 and definition 2 we get

$$(x^k)^k = x^k \quad \text{for} \quad x \in X(A,b),\ \ k \in K(x)\ , \tag{16}$$

$$X_1 = \{x \in X(A,b) \mid x^k = x \text{ for certain } k \in K(x)\}\ , \tag{17}$$

$$X_1 \subset X(A,b) \cap L_0^n\ . \tag{18}$$

Corollary 2. $X_0 \subset X_1$ and

$$x^k = x \quad \text{for} \quad x \in X_0\ , \quad k \in K(x)\ . \tag{19}$$

Proof. If x is a minimal element of X(A,b) then by definition 1 and lemma 3 (cf.(5) and (14)) we get $x^k = x$ for any $k \in K(x)$, i.e. $x \in X_1$ and we get (19).

Corollary 3 (cf. [3]). X_0 is the set of all minimal elements in X_1.

Proof. By above corolaries all elements of X_0 are minimal in X_1. Let suppose that $z \in X_1$ is a minimal element of X_1, and $z \notin X_0$, i.e. there exists $x \in X(A,b)$ such that $x < z$. Then by lemma 3

$$x^k \leqslant x < z \quad \text{and} \quad x^k \in X_1 \quad \text{for } k \in K(x) ,$$

which is contradictory with definition 1. Therefore the only minimal elements of X_1 are elements of X_0.

Corollary 4. Sets X_0 and X_1 are finite.

Lemma 4. For any $x \in X(A,b)$ there exists $z \in X_0$ such that $z \leqslant x$.

Proof. In finite poset any element is bounded below by minimal one (cf. [1], p.4-5). Since any solution $x \in X(A,b)$ is bounded below by its projections from finite set X_1 (cf. lemma 3 and corollary 4), then any solution x is bounded below by certain minimal element $z \in X_1$, which is an element of X_0 (cf. corollary 3).

As immediate consequence of lemmas 1 and 4 we write (cf. [2], [3])

Theorem 2. Let u denote solution (2), then

$$X(A,b) = \{x \in L^n \mid z \leqslant x \leqslant u \quad \text{for certain} \quad z \in X_0\} = \bigcup_{z \in X_0} [z,u].$$

Above characterization reduces problem of determination of infinite family of all solutions to the determination of finite set X_0.

4. Characterization of X_0. Our assumption $X(A,b) \neq \emptyset$ implies that $X_0 \neq \emptyset$ (cf. lemma 4). We reprove (cf. [3])

Lemma 5. Let $x, z \in X(A,b)$, then

$$(x \leqslant z) \Rightarrow (K(x) \subset K(z)) , \tag{20}$$

$$(x \leqslant z) \Rightarrow (x^k = z^k \quad \text{for} \quad k \in K(x)) . \tag{21}$$

Proof. If $x \leqslant z$ and $k \in K(x)$ then from (10) we get

$$b_j = \min(x_{k_j}, a_{k_j,j}) \leqslant \min(z_{k_j}, a_{k_j,j}) \leqslant \max_{i \in I} \min(z_i, a_{ij}) = b_j$$

and $k \in K(z)$ by definition 2, which proves (20). Now any $k \in K(x)$ gives projection (cf. definition 2)

$$x^k = z^k = k(b) ,$$

which proves (21).

Because of corollary 2 (cf.(19) and (21), cf.also [3]) we have

Corollary 5. For any $z \in X_0$ there exists $k \in K(u)$ such that $u^k = z$, where u denotes solution (2).

Now we give a characterization of X_0 different from that of corollary 3(cf.[3])

Theorem 3. Set X_0 can be descripted as

$$X_0 = \{x \in X(A,b) \mid x = x^k \text{ for any } k \in K(x)\} . \tag{22}$$

Proof. After corollary 2 any $x \in X_0$ fulfils (19). Let suppose that an element $x \in X(A,b)$ fulfils (19) and $x \notin X_0$, i.e. there exists $z \in X(A,b)$, such that $z < x$. Then by lemma 3 there exists $k \in K(x)$ such that

$$x^k = z^k \leqslant z < x \quad ,$$

which is contradictory to (19). So we have (22).

5. Determination of X_0. Observe that numeration of rows in system (1) can be permuted such that

$$b_n \leqslant \dots \leqslant b_2 \leqslant b_1 . \tag{23}$$

Under this assumption, for any $z \in X(A,b)$ we form a binary matrix $B = B(z)$,

$$b_{ij} = \begin{cases} 1 , & \text{if } \min(z_i, a_{ij}) = b_j \\ 0 , & \text{if } \min(z_i, a_{ij}) < b_j \end{cases} \quad \text{for } i \in I, \quad j \in J . \tag{24}$$

Rows of this matrix will be denoted by

$$b^i = [b_{i1}, \dots, b_{in}] \quad \text{for } i \in I .$$

Corollary 6. After (10) we get the equivalence

$$(b_{ij} = 1) \Leftrightarrow \exists_{k \in K(z)} (i = k_j) \quad \text{for } i \in I, j \in J \tag{25}$$

We put the following modification of the algorithm from [2]:

Algorithm. Step 0. $s := 1$, $K := \emptyset$, $a_j := 0$ for $j \in J$, $a := [a_1, \dots, a_n]$.

Step 1. Choose an index $k_s \in I$ such that

$$b_{k_s,s} = 1 . \tag{26}$$

Step 2. $a := \max(a, b^{k_s})$,

$$x_{k_s} := b_s \; , \quad K := K \cup \{k_s\} , \tag{27}$$

$$k_j := k_s \text{ if } b_{k_s,j} = 1 \text{ for } j \in J, \; j > s . \tag{28}$$

Step 3. If $P := \{j \in J \mid a_j = 0\} \neq \emptyset$, then $s := \max P$ and go back to Step 1. (29)

Step 4. $x_i := 0$ for $i \in I \setminus K$ (end).

By $X(z)$ we denote set of all $x \in L^m$ obtained from this algorithm. together with $x \in L^m$ also $k \in I^n$ is determined. We can omit (28) if we are interested in determination of x only. Similarly in determination of k, both (27) and (29) can be omitted.

Lemma 6. Let $z \in X(A,b)$. If $x \in L^m$ and $k \in I^n$ are determined by the algorithm, then

$$k \in K(z),\ x = z^k = x^k . \tag{30}$$

Proof. By (26) and (28) $b_{k_s,s} = 1$ for $s \in J$, and corollary 6 implies that $k \in K(z)$ (definition 2).

Under assumption (23) projection (12) reduces to the form $z_i^k = b_s$ with $s = \min J_k(i)$ for $i \in I_k$, and $z_i^k = 0$ for $i \notin I_k$, thus after (27)-(29) $x_i = z_i^k$ for $i \in I$, i.e. $x = z^k$. Now by lemma 3 $k \in K(z^k) = K(x)$ and from lemma 3 we get the right side of (30).

Corollary 7. $X(z) \subset X_1$ for any $z \in X(A,b)$.

Lemma 7. $X(z)$ contains all minimal solutions not greater then z_x for $z \in X(a,b)$.

Proof. Let $z \in X(A,b)$, $v \in X_0$ and $v \leq z$. After corollary 3 and lemma 5 there exists $h \in K(v)$ such that $v = v^h = z^h$. We show that $x = v$ can be obtained in the algorithm for $B = B(z)$.

Since $v = v^h$, then for any $i \in I_h$ there exists $j \in J$ such that $i = h_j$, i.e. (cf. lemma 1 and (24)) $b_{ij} = 1$. Putting $s = \min J_h(i)$ we get

$$v_i = v_i^k = \max_{j \in J_h(i)} b_j = b_s\ ,\quad b_{is} = 1 \tag{31}$$

according to (23). So $i = h_1$ can be choosen in the Step 1 of the algorithm for $s = 1$ and we get

$$k_1 = h_1,\quad x_i = b_1 = v_i,\quad K = \{h_1\}\ ,\quad a = b^{h_1}\ .$$

Successive repetitions in algorithm brings

$$k_s = h_s\ ,\quad x_{k_s} = v_{k_s}\quad \text{for } s \in J$$

according to (27), (28) and (31). Thus algorithm gives $x_i = v_i$ for $i \in K$.

If $i \notin K$, then after (29) $x_i = 0 \leqslant v_i$ and therefore $x \leqslant v$, but v is a minimal solution, whence $x = v$. This proves that $v \in X(z)$.

As a direct consequence for $z = u$ we get

Theorem 4. Let u denote solution (2), then $X_0 \subset X(u)$.

We also get new characterizations of minimal elements

Theorem 5. $X_0 = \{v \in X(A,b) \mid K(v) = \{v\}\}$.

Proof. After theorem 3, for any $v \in X_0$, $K(v)$ contains only v. Now by lemma 7 for any $z \in X(A,b)$, $K(z)$ contains at least one minimal element, which finishes the proof.

Theorem 6. X_0 is the set of all minimal elements in $X(u)$.

Proof. Elements of X_0 are minimal in $X(u)$ because of theorem 4 and corollary 7. Now by lemmas 4 and 7 another minimal element does not exists in $X(u)$.

Example 3. In our example 1 we make suitable permutation. Now

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 7 & 6 & 3 & 3 \\ 4 & 4 & 2 & 0 \end{bmatrix}, \quad b = [4, 4, 3, 3],$$

and condition (23) is valid. For the solutions u, y and z from example 1 we get

$$B(u) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad B(y) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B(z) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Application of algorithm gives $X(u) = \{x^1, x^2, y, z\}$, $X(y) = \{y\}$, $X(z) = \{z\}$, with notmation from example 2. Similarly we get $X(x^1) = \{x^1, y\}$, $X(x^2) = \{x^2, z\}$. We see, that minimal elements y and z can be distinquish in $X(u)$ by theorem 5.

Now let us observe that permutation of columns of matrix A is not unique because $b_1 = b_2$ and $b_3 = b_4$. Using matrix $B(u)$ we can modify first permutation putting the following additional condition

$$(b_j = b_{j+1}) \Rightarrow \left(\sum_{i=1}^{m} b_{ij} \leqslant \sum_{i=1}^{m} b_{i,j+1} \right) \quad \text{for } j \in J . \tag{32}$$

Above used permutation not fulfils condition (32) which leads to the following modification

$$A = \begin{bmatrix} 2 & 4 & 1 & 3 \\ 6 & 7 & 3 & 3 \\ 4 & 4 & 2 & 0 \end{bmatrix}, \quad b = [4,4,3,3], \quad B(u) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Now algorithm gives exactly two solutions: y and z.

6. Conclusion. Presented results allow us to determine all solutions of considered system of max-min equations (1). First, using theorem 1 we verify if $X(A,b) \neq \emptyset$. Next, using solution (2), algorithm produces $X(u)$. Minimal solutions can be distinguish in $X(u)$ by application of theorem 6 (compare any two elements and omit greater element if exists), or theorem 5 (using double algorithm for successive results of the first algorithm omit such results v, which leads to $K(v) \neq \{v\}$). After determination of X_0, set of all solutions is described by theorem 2.

Our consideration can be exactly repeated for dual system of min-max equations

$$\min_{i \in I} \max(x_i, a_{ij}) = b_j \quad \text{for} \quad j \in J . \tag{1'}$$

After transposition of "min" and "max" also transpositions of "$\leqslant$" and "$\geqslant$", "$<$" and "$>$", "0" and "1", "minimal element" and "maximal element", "the greatest element" and "the least element" are necessary. After this transpositions all dual results are valid, and dual algorithm produces maximal solutions of system (1') (cf. definition 1).

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