AN APPROACH TO PROBABILISTIC L-VALUED LOGIC

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Abstract

In this paper an approach to so called probabilistic L-valued logic system constructed by the use of Hirota's concept of probabilistic set is considered.

Some basic notions of L-valued logic are discussed at first and then after a short introduction of the concept of probabilistic set and its distribution function description the fundamental notions of probabilistic L-valued logic are presented.

The considerations are illustrated by means of numerical example.

Key words: Fuzzy set, probabilistic set, distribution function, multiple-valued logic.

1. Introduction

In the two-valued logic it is assumed that every proposition (variable) is either true or false. The formulas (respectively formed expressions) have also two truth-values. In fuzzy logic based on the concepts of fuzzy set and symbolic logic being a special kind of continuous multiple-valued logic it can be assumed for truth value of a formula any value in the interval $[0,1]$ or in a lattice of respective fuzzy subsets.

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The algebraic model of a fuzzy logic is not the Boolean algebra but the Morgan algebra [5].

More generally, it may be considered a logic system whose truth values belong to a lattice $L=\mathcal{F}[0,1]=\{A \mid A : [0,1]\to[0,1]\}$. The further generalization of such system might lie in using instead of lattice $L=\mathcal{F}[0,1]$, a lattice of probabilistic sets in sense of Hirota [3] i.e. $L=p\mathcal{F}[0,1] = \{A \mid A : [0,1]\times\Omega \to [0,1]\}$.

The paper is organized as following. Section 2 presents a condensed description of L-valued logic system and its main notions.
In Section 3 the concept of probabilistic set in sense of Hirota is reminded and its distribution function description is presented as well [1].
Section 4 presents main notions of probabilistic L-valued logic system.
Considerations are illustrated by means of numerical example.
Conclusion remarks pointing out that two-valued logic, L-valued logic are embedded in the probabilistic L-valued logic are included in Section 5.

2. Basic notions of L-valued logic system

According to [4][5] let us remind some basic notions of L-valued logic system.

Let be given a set of elements $\mathcal{X}$, e.g. a set of information items. Denoting by D a descriptor of an element $x \in \mathcal{X}$, it is possible to consider the proposition " x has D " which will be symbolized by a single letter D. If D is interpreted as a fuzzy descripter, a truth value $T(D)$ will be refered to as a numerical truth-value, e.g. $T(D)=0.7$
Supposing that $T(D)$ is not a point in $[0,1]$ but a fuzzy subset of $[0,1]$ we can treat truth as a linguistic variables.

More generally, we can consider a logic system whose truth values belong to a lattice L. This system is defined as follows:

Definition 2.1

An L-valued logic is called a system $\mathcal{L}=(\mathcal{D}, L, T)$ (1)

where $\mathcal{D}$ is the set of propositions

L is a lattice

T is a map $T : \mathcal{D} \to L$ (2)

which assigns to each proposition $D\in\mathcal{D}$ its truth value $T(D)\in L$

The truth value map T must satisfy the following properties:

$$\begin{aligned} &\text{i} && T(R\cup Q) = T(R)\cup T(Q) = \max\left(T(R), T(Q)\right) \\ &\text{ii} && T(R\cap Q) = T(R)\cap T(Q) = \min\left(T(R), T(Q)\right) \\ &\text{iii} && T(\bar{R}) = \overline{T(R)} = 1 - T(R) \end{aligned} \tag{3}$$

In most cases a Morgan algebra will be used as the truth set L. In connection with linguistic variables one have considered as a natural way to use the truth set as $L=\mathcal{F}([0,1]) = \{A \mid A : [0,1]\to[0,1]\}$. For example, " very true " , " not very false " can be used as fuzzy subset of $[0,1]$ belonging to $[0,1]$. Thus the truth value map is the following

$$T : \mathcal{D} \to [0,1] \tag{4}$$

where properties i,ii,iii hold.

Of course, a description of an element of $\mathcal{X}$ might form a formula i.e. a composite proposition.

In L-valued logic can be defined the notion called " satisfiability " of a formula using a concept of measure of fuzziness[2].

Definition 2.2

A formula $D\in\mathcal{D}$ is said to be valid (inconsistent) in sense of energy measure of fuzziness, if

$$d_n^*(T(D)) \geqslant \frac{1}{2} \quad \left(d_n^*(T(D)) < \frac{1}{2}\right) \tag{5}$$

under all of its assignments where $d_n^*(T(D))$ denotes a normalized energy measure of fuzziness of fuzzy set $T(D)$ defined as the following

$$d_n^*(T(D)) = \frac{1}{\nu_0}\int_{[0,1]} T(D)(t)\ d\nu(t) \tag{6}$$

for finite measure ν on $[0,1]$ $(\nu_0 = \nu[0,1] < \infty)$

A formula $D\in\mathcal{D}$ is said to be invalid (consistent), if D is not valid (it is not inconsistent) in sense of Definition 2.2.

Having defined a literal (a variable or its negation), a clause (a disjunction of literals) and a phrase (a conjunction of literals) one can build up a disjunctive (conjunctive) normal forms of formulas in L-valued logic. While it is possible to give some results for evaluation of validity of formulas for two-valued logic by an exhaustive examination of formulas for all possible assignments, there is necessary to investigate the validity of formulas in L-valued logic in sense of Definition 2.2.

3. The concept of probabilistic set and its distribution function representation

Now we will remind shortly the concept of probabilistic set in the sense of Hirota [3]. Introducing the following terms:
$(\Omega, \mathcal{B}, P)$ is a parameter space (probability space) ;
$(\Omega_c, \mathcal{B}_c) = ([0,1]$, Borel sets$)$ is a characteristic space
we will give a definition of probabilistic set:

Definition 3.1

A probabilistic set A in $[0,1]$ is defined by a defining function

$$A : [0,1] \times \Omega \to \Omega_c \tag{7}$$

where $A(x,\cdot)$ is the $(\mathcal{B}, \mathcal{B}_c)$-measurable function for each fixed $x \in [0,1]$
All probabilistic sets in $[0,1]$ is said to be a family of probabilistic sets and will be denoted by $\mathcal{PF}([0,1])$. Some useful operation in $\mathcal{PF}([0.1])$ can be defined. The most important operation for us are the following: union, intersection and complement i.e. for $A, B \in \mathcal{PF}([0,1])$ we have

$$\begin{aligned} A \cup B(x,\omega) &= \max\{A(x,\omega) \ , \ B(x,\omega)\} \\ A \cap B(x,\omega) &= \min\{A(x,\omega) \ , \ B(x,\omega)\} \\ \bar{A}(x,\omega) &= 1 - A(x,\omega) \end{aligned} \tag{8}$$

From a formal point of view $A(x,\cdot)$ can be treated as a random process, when the following condition is fulfilled

$$\forall x \in [0,1] \ \forall z \in [0,1] \qquad \{\omega : A(x,\omega) < z\} \in \mathcal{B} \tag{9}$$

We may also say that $A(x,\cdot)$ is a random variable for each x fixed. Introducing the inclusion relation $\subseteq$ in $\mathcal{PF}([0,1])$ it is shown that $(\mathcal{PF}([0,1]), \subseteq)$ constitutes a partially ordered set [3]. In $(\mathcal{PF}([0,1]), \subseteq)$ the following laws: distributive , de Morgan's, commutative and associative hold. As in fuzzy set theory there is here a lack of complemented laws.

Now we will introduce a distribution function description of a defining function of probabilistic set. Considering now a multidimensional distribution function for any set of numbers $x_1, x_2, \ldots, x_n \in [0,1]$, where the number n is chosen arbitrarily, we have

$$F_{A(x_1)\,A(x_2)\,\ldots\,A(x_n)}(z_1, z_2, \ldots, z_n) = \\ = P(\{\omega : A(x_1,\omega) < z_1, \ A(x_2,\omega) < z_2, \ldots, A(x_n,\omega) < z_n\}) \tag{10}$$

The distribution function must satisfy the following two conditions.

1 The symmetry condition: the equality

$$F_{A(x_{i_1})A(x_{i_2})\dots A(x_{i_n})}(z_{i_1},z_{i_2},\dots,z_{i_n}) =$$

$$= F_{A(x_1)A(x_2)\dots A(x_n)}(z_1,z_2,\dots z_n) \tag{11}$$

holds for any permutation $i_1,i_2,\dots,i_n$ of numbers $1,2,\dots,n$

2 The compatibility condition: if $m<n$, then

$$F_{A(x_1)A(x_2)\dots A(x_n)}(z_1,z_2,\dots,z_m,+\infty,\dots,+\infty) =$$

$$= F_{A(x_1)A(x_2)\dots A(x_m)}(z_1,z_2,\dots,z_m) \tag{12}$$

Because of the importance of max and min functions of probabilistic sets in further considerations we derive their distribution functions. For this purpuse the following theorem holds true[1]

Theorem 3.1

Let $X_1,X_2,\dots,X_n$ are probabilistic sets of the universe of discourse $[0,1]$. If $X_1,X_2,\dots,X_n$ and their respective collections are characterized by distribution functions , then the distribution functions of $\max(X_1,X_2,\dots,X_n)$ and $\min(X_1,X_2,\dots,X_n)$ take the forms

$$F_{\max(X_1,X_2,\dots,X_n)(x)}(w) = F_{X_1(x)X_2(x)\dots X_n(x)}(w,w,\dots,w) \tag{13}$$

$$F_{\min(X_1,X_2,\dots,X_n)(x)}(w)=\sum_{j=1}^{n} F_{X_j(x)}(w) - \sum_{1\le j<k\le n} F_{X_j(x)X_k(x)}(w,w) + \dots + (-1)^{n+1} F_{X_1(x)X_2(x)\dots X_n(x)}(w,w,\dots,w) \tag{14}$$

Assuming additionally the independency of the whole collections of X_1 for each $x\in\mathcal{X}$, the distribution functions of max and min functions can be rewritten as follows

$$F_{\max(X_1,X_2,\dots,X_n)(x)}(w) = \sum_{j=1}^{n} F_{X_j(x)}(w) \tag{15}$$

$$F_{\min(X_1,X_2,\dots,X_n)(x)}(w) = 1 - \sum_{j=1}^{n} \left[1 - F_{X_j(x)}(w)\right] \tag{16}$$

The proof of that theorem can be find in [1].

It is also necessary to have a distribution function of a complement of probabilistic set. For that we will prove the next theorem

Theorem 3.2

If A is a probabilistic set and $\bar{A}$ denotes complement of A, then the distribution function of $\bar{A}$ has a form

$$F_{\bar{A}(x)}(w) = 1 - F_{A(x)}(1 - w) - P(\{\omega : A(x,\omega) = 1 - w\}) \qquad (17)$$

Proof.

Using the obvious equality

$$P(\{\omega : \bar{A}(x,\omega) < w\}) + P(\{\omega : \bar{A}(x,\omega) \geqslant w\}) = 1$$

we get

$$F_{\bar{A}(x)}(w) = P(\{\omega : \bar{A}(x,\omega) < w\}) = 1 - P(\{\omega : \bar{A}(x,\omega) > w\}) - P(\{\omega : \bar{A}(x,\omega) = w\})$$

$$F_{\bar{A}(x)}(w) = 1 - P(\{\omega : 1 - w > A(x,\omega)\}) - P(\{\omega : A(x,\omega) = 1 - w\})$$

Hence

$$F_{\bar{A}(x)}(w) = 1 - F_{A(x)}(1 - w) - P(\{\omega : A(x,\omega) = 1 - w\}) \qquad \text{Q.E.D.}$$

4. The fundamental notions and definitions of probabilistic L-valued logic

Let us assume that a descriptor of an element $x \in \mathscr{X}$ will be represented now by the probabilistic set D such as introduced in Section 3 and will be called a probabilistic descriptor. Suppose that $T(D)$ is neither a point in $[0,1]$, nor a fuzzy subset of $[0,1]$ but it will be represented by a probabilistic set as well. We can consider further generalization of the logic system shortly presented in Section2. whose truth values belong to a lattice of probabilistic sets.

Now we will introduce the following definition of probabilistic L-valued logic system.

Definition 4.1

A probabilistic L-valued logic will be called a system

$$\mathcal{PL} = (\mathcal{D}, L, T) \qquad (18)$$

where $\mathcal{D}$ denotes a set of propositions

$$L = \mathcal{PF}([0,1]) = \{A \mid A : [0,1] \times \Omega \to [0,1]\} \qquad (19)$$

is a lattice of probabilistic sets

T is a map

$$T : \mathcal{D} \to L \qquad (20)$$

The truth value map T satisfies the properties i,ii,iii from Section 2 i.e.

$$\begin{aligned} &\text{i} && T(R\cup Q) = T(R)\cup T(Q) \\ &\text{ii} && T(R\cap Q) = T(R)\cap T(Q) \\ &\text{iii} && T(\bar{R}) = \bar{T}(R) = 1 - T(R) \end{aligned} \tag{21}$$

As the elements belonging to lattice $L = \mathcal{PF}([0,1])$ can be considered such linguistic variables as " probably very false ", " probably very true " etc.

Taking into account a distribution function description of all probabilistic sets truth values we obtain the following expressions:

$$\begin{aligned} &\text{i} && F_{T(R\cup Q)}(w) = F_{T(R)\,T(Q)}(w,w) \\ &\text{ii} && F_{T(R\cap Q)}(w) = F_{T(R)}(w) + F_{T(Q)}(w) - F_{T(R)\,T(Q)}(w,w) \\ &\text{iii} && F_{T(\bar{R})}(w) = F_{\bar{T}(R)}(w) = 1 - F_{T(R)}(1-w) + P(\{\omega : T(R) = 1-w\}) \end{aligned} \tag{22}$$

We can also try to define in probabilistic L-valued logic an analogical concept to this one that is called " satisfiability " in L-valued logic. For this reason we will use the concept of measure of fuzziness of probabilistic set [2]. We can use, for example, the energy measure of fuzziness. Denoting the energy measure of $T(D)$ which is a random variable by $d^*(T(D))$ we can introduce the following definition:

<u>Definition 4.2</u>

A formula $D \in \mathcal{D}$ is said to be probabilistic valid (probabilistic inconsistent) in sense of energy measure of fuzziness, if

$$d_n^*(T(D))(\omega) \geqslant \tfrac{1}{2} \quad \left(d_n^*(T(D))(\omega) \leqslant \tfrac{1}{2}\right) \quad \text{a. e.} \quad \omega \in \Omega \tag{23}$$

or in sense of mean value of probabilistic set $T(D)$, if

$$d_n^*(E(T(D))) \geqslant \tfrac{1}{2} \quad \left(d_n^*(E(T(D))) \leqslant \tfrac{1}{2}\right) \tag{24}$$

We can also build up other criteria for estimation of probabilistic validity (probabilistic inconsistency) using the distribution function description of probabilistic set $T(D)$. Of course a formula $D \in \mathcal{D}$ is said to be probabilistic invalid (consistent) if D is not probabilistic valid (it is not probabilistic inconsistent).

We can also introduce as in Section 2, such concept as:

probabilistic literal
probabilistic clause
probabilistic phrase

where the variables are represented by respective probabilistic sets.

It is also possible to introduce in probabilistic L-valued logic, as it is done in L-valued logic, the disjunctive (conjunctive) normal forms of formulas because of the existence of distributive laws and de Morgan's laws in a lattice of probabilistic sets. We can see that there is no systactical difference between formulas in two-valued logic, in L-valued logic and in L-valued logic as well.
However, taking into account the distribution function description of probabilistic sets it should be noted that there is a lack here the distributivity between max and min functions of probabilistic sets described by means of distribution function (let us note that algebraic (probabilistic) sum (a+b−ab) and product (ab) are not distributive).
As an example let us consider for simplicity, uniformly distributed probabilistic sets R, Q, and S of [0,1] as in Fig. 1, connected in the rule $D = (R \cup Q) \cap \bar{S}$ creating a probabilistic descriptor D
let us determine the validity of D in sense of mean value of all probabilistic sets.

Taking into account that

$$T(D)(t) = \min \left\{ \max \left[T(R)(t),\ T(Q)(t) \right],\ 1 - T(S)(t) \right\} \qquad \forall t \in [0,1]$$

after some calculations we get

$$d_n^*\left(E\left(T(D)\right)\right) = \frac{1}{8} \leq \frac{1}{2}$$

It denotes that T(D) is not probabilistic valid in sense of mean value taking into account an energy measure of fuzziness.

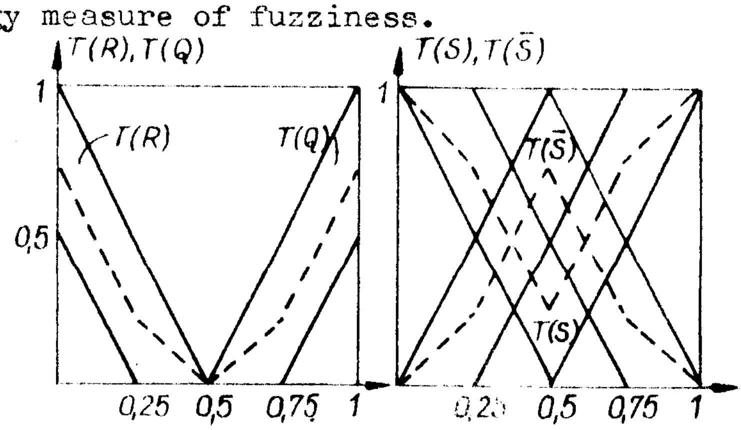

Fig 1

5. Conclusion

In this paper it has been pointed out that two-valued logic and L-valued logic are embedded in the probabilistic L-valued logic, what results from the embedding of crisp set and fuzzy set in probabilistic set.

Basing on presented notions of probabilistic L-valued logic it is possible to characterize some other notions e.g.

$$T(R \to Q) = \min\left(1,\ 1 - T(R) + T(Q)\right)$$

where $T(R)$, $T(Q)$ are probabilistic sets

There is possible to obtain some interesting results for validity (inconsistency) of formulas using the distribution function description of probabilistic sets and the concept of energy measure of fuzziness but it should be a subject of further investigations.

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