A NEW APPROACH TO THE FUZZY CARDINALITY OF FINITE FUZZY SUBSETS[§]

Maciej Wygralak

Institute of Mathematics, A.Mickiewicz University,
60-769 Poznań, Matejki 48/49, Poland

1. Notation

Let U denote an universal set, N denote the set of natural numbers (including zero), A be a finite fuzzy subset of U. Moreover, let $I:=[0,1]$, $I_0:=(0,1]$, $A_t:=\{x: A(x)\geqslant t\}$, where := stands for "equals by the definition". The usual cardinality of a crisp subset Z will be denoted by cardZ.

Let us sort positive membership grades in A in descending order $a_1\geqslant a_2\geqslant\ldots\geqslant a_n$, where n=cardsuppA, and add $a_0:=1$, $a_j:=0$ for $j=n+1,n+2,\ldots$. If A is empty (n=0), then the sequence collapses to $a_0>a_1=a_2=a_3=\ldots$, i.e. $1>0=0=\ldots$.

Fuzzy subsets of N will be called fn-numbers(fuzzy natural numbers). Let P denote a fn-number. The following notation will be used:

$$P=(g_0,g_1,\ldots,g_m),$$

where g_i $(0\leqslant i\leqslant m)$ denotes the membership grade of i in the fn-number P, assuming that membership grades of $m+1,m+2,\ldots$ equal zero.

2. Fuzzy cardinality as a fn-number

First definition in which fuzzy cardinality of A is in the form of a fn-number was introduced by Zadeh in /3/:

$$\mathrm{BFGCount}_A(k):=\max\{t\in I_0:\ \mathrm{card}A_t=k\}.$$

[§]This is abstract of a section of the paper "On fuzzy cardinalities and fuzzy binomial coefficient"(submitted to Fuzzy Sets and Systems Journal).

One can consider $BFGCount_A(k)$ to be degree to which cardinality of A equals k. $BFGCount_A$ is examined with great care in /1/. Unfortunately, the additivity property

$$BFGCount_A \oplus BFGCount_B = BFGCount_{A\cap B} \oplus BFGCount_{A\cup B} ,$$

where $\oplus$ denotes the max-min addition of finite fn-numbers and

$$(P \oplus Q)(m) := \max_{i+j=m} \min(P(i), Q(j)),$$

does not hold.

Another definition of fuzzy cardinality of A was given e.g. in /4/: $FGCount_A(k) := \max\{t \in I_0 : \text{card}A_t \geqslant k\}$. The following propositions are then valid:

(i) $FGCount_A(k) = \max_{l \geqslant k} BFGCount_A(l)$.

(ii) $FGCount_A = (a_0, a_1, a_2, \ldots, a_n)$. Hence $FGCount_A$ is convex.

(iii) $A \subset B$ implies $FGCount_A \subset FGCount_B$.

(iv) $FGCount_A \oplus FGCount_B = FGCount_{A\cap B} \oplus FGCount_{A\cup B}$.

Using the formula (ii) we get for a crisp m-element subset Z of U:

$$FGCount_Z = (1, 1, \ldots, 1) \text{ with } \text{cardsupp}FGCount_Z = m+1.$$

This result is, however, explicable. It suffices to assume that $FGCount_{\bullet}(k)$ measures the degree to which a finite fuzzy subset has at least rather than exactly k elements.

D.Dubois proposed in /2/ another approach to fuzzy cardinality. Let $S_k(A) := \{Y : Y \text{ is crisp, } \text{card}Y = k,\ A_1 \subset Y\}$. Then fuzzy cardinality of A is defined by means of a fn-number Crd_A, where

$$Crd_A(k) := (0 \text{ if } S_k(A) \text{ is empty, else } \max_{Y \in S_k(A)} \min_{x \in Y} A(x))$$

with the understanding that $\min_{x \in \emptyset} A(x) := 1$ (because $S_0(A) = \{\emptyset\}$ if A_1 is empty). Basic properties of Crd_A are listed below:

(i) $Crd_A=(0,\dots,0,1,a_{m+1},a_{m+2},\dots,a_n)$, where $\mathrm{card}A_1=m$. The constant sequence of zeros consists of m elements. Obviously Crd_A is convex.

(ii) $Crd_A=FGCount_A$ if $\mathrm{card}A_1=0$.

(iii) $Crd_A \oplus Crd_B=Crd_{A\cap B} \oplus Crd_{A\cup B}$.

3. A new definition of fuzzy cardinality

Let $P_k(A):=\{Y\colon Y \text{ is crisp}, Y\subset \mathrm{supp}A,\ \mathrm{card}Y=k\}$. A finite fn-number Cd_A such that $Cd_A(k):=(0$ if $P_k(A)$ is empty, else $\max\limits_{Y\in P_k(A)} \mathrm{dist}(A,Y))$,

where (for $p,q\in I$) $p\to q:=\min(1,1-p+q)$ (Łukasiewicz implication operator),

$$p\leftrightarrow q:=\min(p\to q, q\to p),$$

$$\mathrm{dist}(A,B):=\min_{x\in U}(A(x)\leftrightarrow B(x)),$$

will be called Cd-cardinality of A. One can consider $Cd_A(k)$ to be degree to which A has exactly k elements. To be exact, $Cd_A(k)$ is a quality of the best approximation of A (using the criterion dist(A,)) by means of k-element crisp subsets of the support of A.

The following properties of the Cd-cardinality hold:

(i) $Cd_A(k)=\min(a_k,1-a_{k+1})=\min(FGCount_A(k),1-FGCount_A(k+1))$.

(ii) $Cd_A(k)=1$ iff $k=n$ and A is crisp.

$Cd_A(k)=0$ iff $k>n$ or ($k<n$ and $\mathrm{card}A_1\geqslant k+1$).

(iii) $Cd_A=(1-a_1,1-a_2,\dots,1-a_p,a_p,a_{p+1},\dots,a_n)$, where $p:=\min\{l\colon a_l+a_{l+1}\leqslant 1\}$. Hence Cd_A is convex.

(iv) It exists at most one natural number $\underline{k}$ such that $Cd_A(\underline{k})>0.5$.

(v) $Cd_A= 2Cd_{0.5A}\cap 2Cd_{0.5A+0.5}$, where $(cF)(x):=\min(1,cF(x))$ for a finite fuzzy subset F ($c>0$) and $(0.5A+0.5)(x):=0.5A(x)+0.5$.

(vi) $FGCount_A= 2Cd_{0.5A}$.

(vii) $Cd_A \oplus Cd_B = Cd_{A\cap B} \oplus Cd_{A\cup B}$.

(viii) If $\mathrm{card}U=m<\infty$, then $Cd_{A^c}(j)=Cd_A(m-j)$ for $j=0,1,\dots,m$ (A^c denotes complement of A).

Properties (iv) and (viii) do not hold for both $FGCount_A$ and Crd_A. Property (iv) says that the Cd-cardinality of A "favours" at most one cardinal number. Property (viii) is a fuzzy counterpart of the equality $cardZ^c = m-cardZ$, where Z denotes a crisp subset of a finite m-element universal set.

References

/1/ D.Dubois, Propriétés de la cardinalité floue d'un ensemble flou fini, BUSEFAL, 5, 11-12, 1981.

/2/ D.Dubois, A new definition of the fuzzy cardinality of finite fuzzy sets preserving the classical additivity property, BUSEFAL, 8, 65-67, 1981.

/3/ L.A.Zadeh, A theory of approximate reasoning, in: J.Hayes, D.Michie, L.Mikulich, Eds., Machine Intelligence, vol.9 (John Wiley and Sons, 1979), 149-194.

/4/ L.A.Zadeh, Fuzzy probabilities and their role in decision analysis, in: Proc. IFAC Symp. on Th. and Appl. of Digital Control, New Dehli (January, 1982).