Distribution of Fuzzy Subsets Projected from Random Intervals

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Abstract

In this paper the computational formulas of membership function of $\underset{\smile}{s}$ projected from random intervals has been proved on the basis of [1] , and the properties of $\underset{\smile}{s}$ has been discussed.

1. Preparatory Knowledge

Wang Pei-zhuang and E. Sanchez studied relation between fuzzy subsets and random subsets in [1] . This paper is based on [1]. The definitions and theorem of [1] relating to the paper are given as preparatory knowledge as follows.

$S: \Omega \rightarrow \mathcal{B}$, is a projectable random subset of U.

$\mathcal{B}_\bullet$ is the Borel field on $\mathbb{R}$.

$\mathcal{B}_\bullet^2$ is the Borel field on $\mathbb{R}^2$.

$\delta = \{ [X_1, X_2] \mid X_1, X_2 \in \mathbb{R}, X_1 \leq X_2 \}$, $[X_1, X_2]$ is a closed interval.

$\underset{\smile}{S}$ is a fuzzy subset projected from S .

$R^2/ = \{(x, y) \mid x, y \in \mathbb{R}, \quad x \leq y\}$, (x, y) is a Point in $\mathbb{R}^2$

$\mathcal{B}^2/ = \{D \mid D \in \mathcal{B}_0^2, \quad D \subset R^2/\}$

$V = \{\xi \mid \xi : \Omega \to R^2/ \quad , \xi \text{ is } \mathcal{A}\text{-}\mathcal{B}^2/ \text{ measurable}\}$

$\psi : V \to \delta^{\Omega}$

$\lrcorner_x = (-\infty, x] \times [x, +\infty)$

Preparatory Theorem. Suppose that $\xi = (\xi_1, \xi_2) \in V$ and $S = \psi(\xi)$, then we have $\mu_{S}(x) = P(\xi \in \lrcorner_x)$

2. The computational formulas of membership function of S projected from random intervals [1]

The preparatory theorem mentioned above gives out the relation between projectable random intervals $S = \psi(\xi)$ and random variable $\xi = (\xi_1, \xi_2) \in V$. If ξ has density $p(x_1, x_2)$, then

$$\mu_{S}(x) = \int_x^{+\infty} \left(\int_{-\infty}^{x} p(x_1, x_2)\, dx_1 \right) dx_2 \qquad \text{(See [1])}$$

In general, we have the following results.

<u>Theorem 1:</u> Suppose that $\xi = (\xi_1, \xi_2) \in V$ and ξ has distribution function $F(x_1, x_2)$, let $S = \psi(\xi)$, then we have that

$$\mu_{S}(x) = F(x+0, +\infty) - F(x+0, x)$$

Proof: According to the preparatory theorem we have that

$$\begin{aligned} \mu_{S}(x) &= P(\xi \in \lrcorner_x) \\ &= P(\xi \in (-\infty, x] \times [x, +\infty)) \\ &= P(\xi_1 \in (-\infty, x], \ \xi_2 \in [x, +\infty)) \end{aligned}$$

$$= P(\xi_1 \le x,\ \xi_2 < +\infty) - P(\xi_1 \le x,\ \xi_2 < x)$$

$$= F(x+0, +\infty) - F(x+0,\ x) \qquad \blacksquare$$

N.B. : The previous distribution function is defined as $F(x_1, x_2) = P(\xi_1 < x_1,\ \xi_2 < x_2)$.

Corollary 1.1 : Suppose that $\xi = (\xi_1, \xi_2) \in V$ and $S = \psi(\xi)$ If ξ is a continuous random variable having distribution function $F(x_1, x_2)$, then

$$\mu_S(x) = F(x, +\infty) - F(x, x).$$

Corollary 1.2: Suppose that $\xi = (\xi_1, \xi_2) \in V$ and $S = \psi(\xi)$. If ξ_1, ξ_2 are independent continuous random variables having distribution functions $F_1(x)$ and $F_2(x)$, then

$$\mu_S(x) = F_1(x)(1 - F_2(x))$$

Corollary 1.3 : Suppose that $\xi = (\xi_1, \xi_2) \in V$ and $S = \psi(\xi)$. If ξ has distribution law $P(\xi = (x_i, y_i)) = p_i$, then

$$\mu_S(x) = \sum_{\substack{x_i \le x \\ y_i \ge x}} p_i$$

Corollary 1.4 : Suppose that $\xi = (\xi_1, \xi_2) \in V$ and $S = \psi(\xi)$ If ξ_1, ξ_2 are independent random variables having distribution laws $P(\xi_1 = x_i) = p_i$ and $P(\xi_2 = y_j) = p_j$, then

$$\mu_S(x) = \Big(\sum_{x_i \le x} p_i\Big) \cdot \Big(\sum_{y_j \ge x} p_j\Big)$$

3. Computational Formulas of $\mu_S(\eta_1, \eta_2)(x)$

Definition: Suppose that η_1, η_2 are independent random variables, let $\xi_1=\min\{\eta_1,\eta_2\}$, $\xi_2=\max\{\eta_1,\eta_2\}$, $\xi=(\xi_1,\xi_2)$ and $S=\psi(\xi)$. We call $\underset{\vee}{S}$ projected from S the fuzzy subset induced by η_1 and η_2, and denote it by $\underset{\vee}{S}(\eta_1,\eta_2)$. In particular, the $\underset{\vee}{S}(\eta_1,\eta_2)$ will be denoted by $\underset{\vee}{S}(\eta)$ if η_1,η_2 are independent same distribution.

Theorem 2: Suppose that η_1,η_2 are independent random variables having distribution functions $F_1(y)$ and $F_2(y)$, then we have that

$$\mu_{\underset{\vee}{S}(\eta_1,\eta_2)}(x)=F_1(x+0)+F_2(x+0)-F_1(x+0)F_2(x+0)-F_1(x)F_2(x)$$

Proof : $\mu_{\underset{\vee}{S}(\eta_1,\eta_2)}(x)=P(\xi\in\perp_x)$

$$\begin{aligned}
&=P(\min\{\eta_1,\eta_2\}\le x,\ \max\{\eta_1,\eta_2\}\ge x)\\
&=P(\min\{\eta_1,\eta_2\}\le x,\ \max\{\eta_1,\eta_2\}<+\infty)\\
&\quad-P(\min\{\eta_1,\eta_2\}\le x,\ \max\{\eta_1,\eta_2\}<x)\\
&=P(\min\{\eta_1,\eta_2\}\le x)-P(\max\{\eta_1,\eta_2\}<x)\\
&=1-P(\eta_1>x)\cdot P(\eta_2>x)-F_1(x)\cdot F_2(x)\\
&=F_1(x+0)+F_2(x+0)-F_1(x+0)\cdot F_2(x+0)-F_1(x)\cdot F_2(x)
\end{aligned}$$

Corollary 2.1: If η_1,η_2 are continuous random variables having distribution functions $F_1(y)$ and $F_2(y)$, then

$$\mu_{\underset{\vee}{S}(\eta_1,\eta_2)}(x)=F_1(x)+F_2(x)-2F_1(x)F_2(x)$$

Corollary 2.2: If η_1,η_2 are independent random variables having distribution laws $P(\eta_1=y_{1i})=P_{1i}$ and $P(\eta_2=y_{2j})=P_{2j}$, then

$$\mu_{\underset{\vee}{S}(\eta_1,\eta_2)}(x)=\sum_{y_{1i}\le x}P_{1i}+\sum_{y_{2j}\le x}P_{2j}-\Big(\sum_{y_{1i}\le x}P_{1i}\Big)\cdot\Big(\sum_{y_{2j}\le x}P_{2j}\Big)-\Big(\sum_{y_{1i}<x}P_{1i}\Big)\Big(\sum_{y_{2j}<x}P_{2j}\Big)$$ ∎

Theorem 3: Suppose that η_1, η_2 are independent continuous random variables having distribution Functions $F_1(y)$ and $F_2(y)$, then $\underset{\smile}{S}(\eta_1, \eta_2)$ is a fuzzy number[2] iff $\exists x_0 \in \mathbb{R}$ so that $F_1(x_0)=0$, $F_2(x_0)=1$ or $F_1(x_0)=1$, $F_2(x_0)=0$.

Proof:

If $\exists x_0 \in \mathbb{R}$ so that $F_1(x_0)=0$, $F_2(x_0)=1$ or $F_1(x_0)=1$, $F_2(x_0)=0$, let $A=\{x_0 \mid F_1(x_0)=0, F_2(x_0)=1 \text{ or } F_1(x_0)=1, F_2(x_0)=0\}$. Since $F_1(x)$, $F_2(x)$ are monotone increasing, we can prove that A is a closed interval. Set $A=[x_{01}, x_{02}]$, where $x_{01} \le x_{02}$.

$\forall x_0 \in A$, $\mu_{\underset{\smile}{S}(\eta_1,\eta_2)}(x_0) = F_1(x_0) + F_2(x_0) - 2F_1(x_0) \cdot F_2(x_0) = 1$. $\forall x_1 < x_2 < x_{01}$,

$$\mu_{\underset{\smile}{S}(\eta_1,\eta_2)}(x_1) - \mu_{\underset{\smile}{S}(\eta_1,\eta_2)}(x_2) =$$
$$\begin{cases} F_2(x_1) - F_2(x_2) \le 0; & F_1(x_{01})=0, F_2(x_{01})=1. \\ F_1(x_1) - F_1(x_2) \le 0; & F_2(x_{01})=0, F_1(x_{01})=1. \end{cases}$$

Thus it follows that if $x < x_{01}$, $\mu_{\underset{\smile}{S}(\eta_1,\eta_2)}(x)$ is monotone increasing. Similarly; if $x > x_{02}$, $\mu_{\underset{\smile}{S}(\eta_1,\eta_2)}(x)$ is monotone decreasing. Hence, $\underset{\smile}{S}(\eta_1, \eta_2)$ is a fuzzy number.

On the contrary, if $\underset{\smile}{S}(\eta_1, \eta_2)$ is a fuzzy number, then $\exists x_0 \in \mathbb{R}$ so that

$\mu_{\underset{\smile}{S}(\eta_1,\eta_2)}(x_0) = F_1(x_0) + F_2(x_0) - 2F_1(x_0)F_2(x_0) = 1$. Since $a+b-2ab=1$ $(0 \le a, b \le 1)$ iff $a=0$, $b=1$ or $a=1$, $b=0$, thus $F_1(x_0)=0$, $F_2(x_0)=1$ or $F_1(x_0)=1$, $F_2(x_0)=0$. ∎

4. Computational Formulas of $\mu_{\underset{\smile}{S}(\eta)}(x)$

Using Theorem 2 and its corollaries, we obtain the theorem and corollaries as follows.

Theorem 4: Suppose that η_1, η_2 are independent random variables having same distribution function $F_\eta(y)$, then we have that

$$\mu_{\underset{\smile}{S}(\eta)}(x) = 2F_\eta(x+0) - F_\eta^2(x+0) - F_\eta^2(x)$$

Corollary 4.1: If η_1, η_2 are independent continuous random variables having same distribution function $F_\eta(y)$, then

$$\mu_{\underset{\smile}{S}(\eta)}(x) = 2F_\eta(x) \cdot \{1 - F_\eta(x)\}$$

Corollary 4.2: If η_1, η_2 are independent random variables having same distribution law $P(\eta = y_i) = p_i$. then

$$\mu_{\underset{\smile}{S}(\eta)}(x) = 2\sum_{y_i \le x} p_i - (\sum_{y_i \le x} p_i)^2 - (\sum_{y_i < x} p_i)^2$$

Proposition. Suppose that η_1, η_2 are independent continuous variables having same distribution function $F_\eta(y)$, then $\underset{\smile}{S}(\eta)$ has following properties:

(1) $\mu_{\underset{\smile}{S}(\eta)}(x)$ is continuous on $\mathbb{R}$.

(2) $\max \mu_{\underset{\smile}{S}(\eta)}(x) = \mu_{\underset{\smile}{S}(\eta)}(x_0) = \frac{1}{2}$, Where $x_0 \in \{x_0 \mid F(x_0) = \frac{1}{2}\}$

(3) $\underset{\smile}{S}(\eta)$ is convex.

Proof: (1). Corollary 4.1 shows that $\mu_{\underset{\smile}{S}(\eta)}(x) = 2F_\eta(x) \cdot \{1 - F_\eta(x)\}$

Since $F_\eta(x)$ is continuous, $\mu_{\underset{\smile}{S}(\eta)}(x)$ is contimuous.

(2) $\mu_{\underset{\smile}{S}(\eta)}(x) = 2F_\eta(x)\{1 - F_\eta(x)\} = -2(F_\eta(x) - \frac{1}{2})^2 + \frac{1}{2}$

Since $F_\eta(-\infty) = 0$ and $F_\eta(+\infty) = 1$, $\{x_0 \mid F_\eta(x_0) = \frac{1}{2}\} \neq \phi$.
Hence $\max \mu_{\underset{\smile}{S}(\eta)}(x) = \mu_{\underset{\smile}{S}(\eta)}(x_0) = \frac{1}{2}$, Where $x_0 \in \{x_0 \mid F(x_0) = \frac{1}{2}\}$.

(3) let $A = \{x_0 \mid F_\eta(x_0) = \frac{1}{2}\}$ Since $F_\eta(x)$ is continuous and monotone increasing, we have that A is a

closed interval . Let $A = [x_{01}, x_{02}]$.

$\forall x_0 \in A$, $\mu_{\underset{\sim}{\xi}(\eta)}(x_0) = \frac{1}{2}$

$\forall x_1 < x_2 < x_{01}$

$$\begin{aligned}\mu_{\underset{\sim}{\xi}(\eta)}(x_1) - \mu_{\underset{\sim}{\xi}(\eta)}(x_2) &= 2F_\eta(x_1)[1-F_\eta(x_1)] - 2F_\eta(x_2)\cdot \\ &\quad [1-F_\eta(x_2)] \\ &= 2[F_\eta(x_1) - F_\eta(x_2)]\cdot[1-(F_\eta(x_1) + \\ &\quad F_\eta(x_2))] \le 0\end{aligned}$$

Thus, if $x < x_{01}$, $\mu_{\underset{\sim}{\xi}(\eta)}(x)$ is monotone increasing.

Similarly, if $x > x_{01}$, $\mu_{\underset{\sim}{\xi}(\eta)}(x)$ is monotone decreasing

Hence $\mu_{\underset{\sim}{\xi}(\eta)}(x)$ is convex. ∎

Finally, we give some figures of $\mu_{\underset{\sim}{\xi}(\eta)}(x)$, where η is random variable having moment probability distribution. (See next page)

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References

[1] Wang Pei-zhuang, Sanchez, E., Treating a fuzzy subset as a projectable random subset, Fuzzy Information and Decision Processes', Edited by M. M. Gupta. E. Sanchez, (1983) PP 213-219.

[2] Luo Cheng-zhong, 模糊集引论 Beijing Normal University (1983)PP 147-151.

Figures of $\mu_{\xi(\eta)}(x)$

<table>
<tr><th>η</th><th>$\mu_{\xi(\eta)}(x)$</th><th>η</th><th>$\mu_{\xi(\eta)}(x)$</th></tr>
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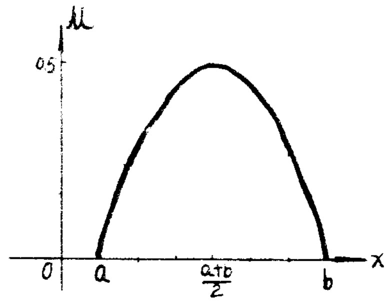
<td>
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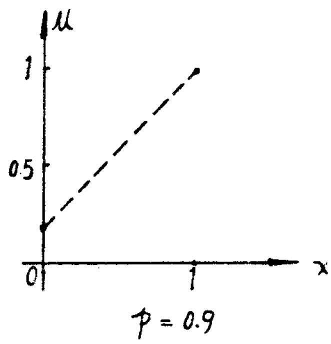
<td>
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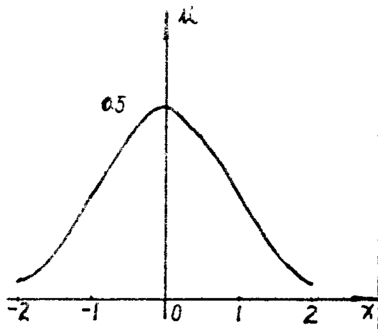
<td>
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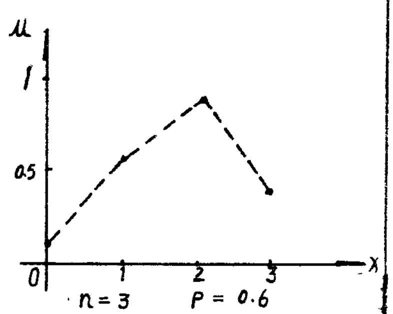
$k=0,1,2\cdots n$</td>
<td>
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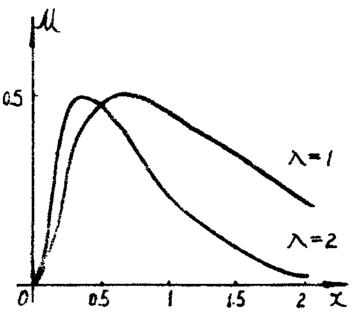
<td>
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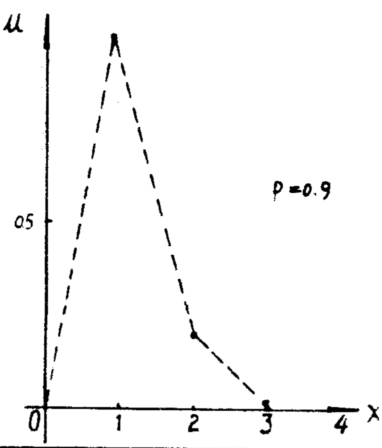
<td>
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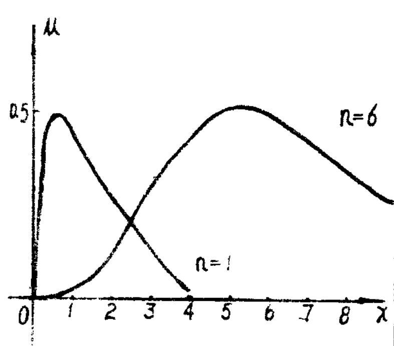
<td>
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$p_x = \frac{\lambda^k}{k!}e^{-\lambda}$
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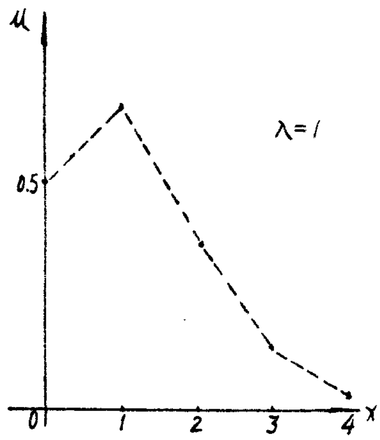
<td>
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