

Fuzzy Mappings and Fixed Point Theorems

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In this paper the problem of the existence of fixed point for fuzzy mappings is approached. At first we introduce a concept of a fuzzy mapping, i.e. mapping from an arbitrary set to one subfamily of fuzzy sets in a metric linear space X . Each element of this family is interpreted as an approximate quantity. Then we prove two fixed point theorems for fuzzy mappings. These theorems is a generalization of S.Heilpern's fixed point theorem.

Keywords: Fuzzy mapping, Fixed point, Fuzzy orbit.

1. Introduction

The fixed point theorems for set-valued mapping play a very important role in classical game theory. The fuzzy fixed point theorems are useful tools for solving various problems of the "theory of noncooperative N-games fuzzy games" [1—4], which form a part of fuzzy analysis. Recently T.Butnariu [5] and S.Heilpern [6] gave several fuzzy fixed point theorems. In the present paper we obtain two fixed point theorems for fuzzy mapping which generalize S.Heilpern's [6] results. The main results are theorem 3.1 and theorem 3.2. Theorem 3.1 improves S.Heilpern's theorem 3.1. The relation between fuzzy orbit and fuzzy fixed point should be an interesting problem. We shall discuss it in theorem 3.2.

2. Preliminaries

This section will present some basic knowledge of fuzzy mapping and

at performance quantity, then give some properties of it. But we shall omit proofs, readers can find them in [6].

Throughout this paper, X is a metric linear space and d is a metric in X . $\mathcal{F}(X)$ is the collection of all fuzzy sets in X . If $A \in \mathcal{F}(X)$, $x \in X$, the function-value $A(x)$ is called the grade of membership of x in A . A_α is the α -level set of A .

Definition 2.1. A fuzzy subset A of X is a general approximate quantity if A_α ($\alpha \in [0,1]$) is a closed convex subset of X for each $\alpha \in [0,1]$ and $\text{Sup}_{x \in X} A(x) = 1$. The collection of all general approximate quantity in X is denoted by $GW(X)$ (the sense of notation $W(X)$ is still as [6]).

Definition 2.2. Let $A, B \in GW(X)$, $\alpha \in [0,1]$. Define

$$P_\alpha(A, B) = \inf_{x \in A_\alpha, y \in B_\alpha} d(x, y),$$

$$D_\alpha(A, B) = H(A_\alpha, B_\alpha),$$

$$D(A, B) = \sup_{\alpha} D_\alpha(A, B).$$

where H is Hausdorff distance. The function P_α is called a α -space, D_α a α -distance and D a distance between A and B .

Definition 2.3. Let $A, B \in GW(X)$. Define

$$A \subset B \iff A(x) \leq B(x) \quad \forall x \in X$$

Proposition 2.4. Let X be an arbitrary set and Y any metric linear space. $F : X \rightarrow Y$ is called a fuzzy mapping iff F is mapping from the set X into $GW(Y)$, i.e., $F(x) \in GW(Y)$, $x \in X$.

Lemma 2.5. Let $x \in X$, $A \in GW(X)$ and $\{x\}$ be a fuzzy set with membership function equal a characteristic function of set $\{x\}$. If $\{x\} \subset A$, then

$$P_\alpha(\{x\}, A) = P_\alpha(x, A), \quad \alpha \in [0,1].$$

$$\text{Proof.} \quad P_\alpha(x, A) \leq d(x, y) + P_\alpha(y, A) \quad x, y \in X.$$

$$\text{Conversely, if } \{x_0\} \subset A, \text{ then } P_\alpha(x_0, B) \leq D_\alpha(A, B), \quad B \in GW(X).$$

3. Fixed Point Theorem for Fuzzy Mappings

In this section we give two fixed point theorems for fuzzy mapping.

Theorem 3.1. Suppose (X, d) is a complete metric linear space, $\{F_i\}$, $i = 1, 2, \dots$ is a sequence of fuzzy mappings of X into $GW(X)$ which satisfies the following conditions:

$$\begin{aligned} d(F_i(x), F_j(y)) &\leq h \max \{d(x, y), P_i(x, F_i(x)), P_i(y, F_j(y)), \\ &\quad \frac{1}{2}[P_i(x, F_j(y)) + P_j(y, F_i(x))]\} \quad x, y \in X, i, j = 1, 2, \dots \end{aligned} \quad (1)$$

where $0 < h < 1$. Then there exists at least one \hat{x} , such that for any

$\{x_n\} \subset F_1(\hat{x})$

$$\{x_n\} \subset F_1(\hat{x}), \quad i = 1, 2, \dots$$

and x_n is the point $\{\hat{x}\}$ a common fixed point of fuzzy mappings F_i , $i = 1, 2, \dots$

Proof. Let $x_0 \in X$ be an arbitrary point and $\{x_1\} \subset F_1(x_0)$. According to Definition of Hausdorff's distance there exists $x_2 \in X$ such that $\{x_2\} \subset F_2(x_1)$ and $d(x_1, x_2) \leq KD_1(F_2(x_1), F_1(x_0))$, where $1 < K < \frac{1}{h}$.

Using exactly the same method we can obtain a sequence $x_n: n = 2, 3, \dots$ such that for any $n = 2, 3, \dots$

$$x_{n+1} \in F_{n+1}(x_n) \quad \text{and} \quad d(x_{n+1}, x_n) \leq KD_1(F_{n+1}(x_n), F_n(x_{n-1}))$$

that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq KD_1(F_{n+1}(x_n), F_n(x_{n-1})) \leq Kh \max \{d(x_n, x_{n-1}), \\ &\quad P_1(x_n, F_{n+1}(x_n)), \frac{1}{2}[P_1(x_{n-1}, F_{n+1}(x_n)) + P_1(x_n, F_n(x_{n-1}))]\} \end{aligned}$$

It follows to know that

$$d(x_{n+1}, x_n) \leq Kh d(x_n, x_{n-1})$$

then

$$d(x_{n+1}, x_n) \leq (Kh)^n d(x_1, x_0)$$

Since X is a complete metric linear space, $Kh < 1$, then $\{x_n: n = 0, 1, 2, \dots\}$

we can get Cauchy's sequence. Let

$$\lim_{n \rightarrow \infty} x_n = \hat{x} \in X$$

Now we consider

$$\begin{aligned} P_1(\hat{x}, F_1(\hat{x})) &\leq d(\hat{x}, x_n) + P_1(x_n, F_1(\hat{x})) \\ &\leq d(\hat{x}, x_n) + D(F_n(x_{n-1}), F_1(\hat{x})) \quad \dots \dots \quad (2) \end{aligned}$$

$$\begin{aligned} D(F_n(x_{n-1}), F_1(\hat{x})) &\leq h \max \{ d(x_{n-1}, \hat{x}), P_1(x_{n-1}, F_n(x_{n-1})), P_1(\hat{x}, F_1(\hat{x})), \\ &\quad \frac{1}{2}[P_1(x_{n-1}, F_1(\hat{x})) + P_1(\hat{x}, F_n(x_{n-1}))] \} \\ &\leq h \max \{ d(x_{n-1}, \hat{x}), P_1(x_{n-1}, x_n), P_1(\hat{x}, F_1(\hat{x})), \\ &\quad \frac{1}{2}[P_1(x_{n-1}, F_1(\hat{x})) + P_1(\hat{x}, x_n)] \} \quad (3) \end{aligned}$$

From (2) and (3), let $n \rightarrow \infty$ we have

$$P_1(\hat{x}, F_i(\hat{x})) \leq h P_1(\hat{x}, F_1(\hat{x}))$$

Since $0 < h < 1$, then $P_1(\hat{x}, F_1(\hat{x})) = 0$. Thus

$$\{\hat{x}\} \subset F_i(\hat{x}) \quad i = 1, 2, \dots$$

Theorem 3.1. Theorem 3.1 generalize S.Heilpern's Theorem 3.1 [6] in following three aspects: At first the contraction condition (1) in this paper is more general than that in [6]. Secondly we consider a common domain and all of a sequence of countable fuzzy mappings F_i , $i = 1, 2, \dots$. Thirdly, $\forall \alpha$, $i = 1, 2, \dots$, S.Heilpern's theorem is at once obtained. At last, the consider approximate quantity is more general. The approximate quantity we demand only that $A_\alpha(\forall \alpha)$ is a closed convex subset of $W(X)$ (not necessarily compact).

Now we consider a fixed point theorem for fuzzy nonextension mapping.

Theorem 3.2. Let $F: X \rightarrow W(X)$ (set of approximate quantity, see [6])

be a fuzzy mapping, $x_0 \in X$, by U7) we can find a $\{x_j\} \subset F(x_0)$ such that

$$d(x_n, x_j) = P_1(x_0, F(x_j))$$

By similar method we can obtain a sequence $\{x_{n+1}\} \subset F(x_n)$, $n=1,2,\dots$. If $P_1(x_n, x_{n+1}) = P_1(x_n, F(x_n))$, $n=1,2,\dots$, then we call the sequence $\{x_n\}$, $n=1,2,\dots$ a fuzzy orbit of fuzzy mapping F at x_0 and write $O_F(F, x_0)$.

Theorem 2.2. Suppose (X, d) is a complete metric linear space, fuzzy mapping $F: X \rightarrow \mathbb{F}(X)$ satisfies the following condition

$$\begin{aligned} D(F(x), F(y)) &\leq \max \{d(x,y), P_1(x, F(x)), P_1(y, F(y)), \\ &\quad \frac{1}{2}[P_1(x, F(y)) + P_1(y, F(x))] \} \end{aligned} \quad (4)$$

Let $x, y \in X$.

If $F: (X, d) \rightarrow (\mathbb{F}(X), D(\cdot, \cdot))$ is continuous. If there exist a point $x_0 \in X$ such that fuzzy orbit $O_F(F, x_0) = \{x_n, n=0,1,2,\dots\}$ has a cluster point \hat{x} . Then \hat{x} is fuzzy fixed point of F , i.e. $\{\hat{x}\} \subset F(\hat{x})$.

Proof. If $\hat{x} \neq x_k$ for some k , $x_k = x_{k+1} = \hat{x}$, then it follows $\{\hat{x}\} \subset F(\hat{x})$ from $x_{k+1} \in O_F(F, x_k)$. Hence the proof of theorem is completed.

Now we prove if $x_n \neq x_{n+1}$, $n=1,2,\dots$. Hence

$$\begin{aligned} P_1(x_n, x_{n+1}) &= P_1(x_n, F(x_n)) \leq D(F(x_{n-1}, F(x_n)) \leq D(F(x_{n-1}, F(x_n))) \\ &\leq \max \{d(x_{n-1}, x_n), P_1(x_{n-1}, F(x_{n-1})), P_1(x_n, F(x_n)), \\ &\quad \frac{1}{2}[P_1(x_{n-1}, F(x_n)) + P_1(x_n, F(x_{n-1}))] \} \end{aligned} \quad n \geq 1$$

As a conclusion of $\{x_n, n=1,2,\dots\}$ we have

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \quad n \geq 1$$

From above we get

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = r \geq 0$$

Since $\{x_n\}$, $n \in \{0,1,2,\dots\}$ has a cluster point \hat{x} , i.e. there exist a subsequence $\{x_{n_i}\}$, $i=1,2,\dots$ such that $x_{n_i} \rightarrow \hat{x}$, as $i \rightarrow \infty$. By Lemma 2.3

and $\hat{x} \in \mathbb{F}(X)$ and $A, B \in CB(X)$, $x \in X$, we have

$$|P_1(x, \hat{x}) - \alpha(x, \hat{x})| \leq H(A, B) \quad (5)$$

$\mathcal{C}(P)$

$$\lim_{i \rightarrow \infty} d(x_{n_i}, x_{n_i+1}) = P_1(\hat{x}, F(\hat{x})) \quad (6)$$

$\mathcal{C}(P)$

$$d(x_{n_i}, x_{n_i+1}) = P_1(x_{n_i}, F(x_{n_i}))$$

$$|d(x_{n_i}, F(x_{n_i+1})) - P_1(x_{n_i}, F(\hat{x}))| \leq D_1(F(x_{n_i}), F(\hat{x})) \leq D(F(x_{n_i}), F(\hat{x}))$$

From the continuity of F and let $i \rightarrow \infty$ then we obtain

$\lim_{i \rightarrow \infty} d(x_{n_i}, F(\hat{x})) = 0$. Hence the proof of (6) is completed.

From (6), $d(x_{n_i+1}, F(\hat{x})) \leq D(F(x_{n_i}), F(\hat{x})) \rightarrow 0$, as $i \rightarrow \infty$ and $F(\hat{x}) \in W(X)$,

there exists $\{x_{n_i}\} \subset F(\hat{x})$ such that

$$d(x_{n_i+1}, \hat{x}) \rightarrow 0 \quad \text{as } i \rightarrow \infty$$

And from the compactness of $(F(\hat{x}))_1$ there exists a subsequence $\{x_{n_{i_k}}\}_{k=1}^{\infty}$

of $\{x_{n_i}\}_{i=1}^{\infty}$ such that

$$\lim_{k \rightarrow \infty} \hat{x}_{n_{i_k}} = \{x^*\} \subset F(\hat{x})$$

Then

$$\lim_{i \rightarrow \infty} d(x_{n_i+1}, x^*) = 0$$

By repeating the same method from the proof of (6) we can obtain

$$\lim_{i \rightarrow \infty} d(x_{n_i+1}, x_{n_i+2}) = P_1(x^*, F(x^*))$$

$\lim_{i \rightarrow \infty} d(x_{n_i+1}, x_{n_i+1}) = 0$ and $\lim_{k \rightarrow \infty} d(x_{n_{i_k}}, x_{n_{i_k}+1}) = d(\hat{x}, x^*)$, so we obtain

$$d(\hat{x}, \hat{x}) = P_1(\hat{x}, F(\hat{x})) = P_1(x^*, F(x^*)) = d(\hat{x}, x^*)$$

$\{x^*\} \subset F(\hat{x})$

$$\{\hat{x}\} \neq \{x^*\} \subset F(\hat{x})$$

Let $\hat{x} \in \hat{F}(\hat{x}, x^*)$, using the condition (4)

$$d(\hat{x}, F(x^*)) \leq D_1(F(\hat{x}), F(x^*)) \leq D(F(\hat{x}), F(x^*))$$

$$\text{and } d(\hat{x}, x^*) = \max \{ d(\hat{x}, x^*), P_1(\hat{x}, F(\hat{x})), P_1(x^*, F(x^*)), \frac{1}{2}[P_1(x^*, F(\hat{x})) + P_1(\hat{x}, F(x^*))] \}$$

If $d(\hat{x}, F(\hat{x})) = 0$, then $P_1(x^*, F(\hat{x})) = 0$ and $P_1(\hat{x}, F(x^*)) \leq d(\hat{x}, x^*) + P_1(x^*, F(x^*))$

$$\begin{aligned} d(\hat{x}, x^*) &= \max \{ d(\hat{x}, x^*), P_1(\hat{x}, F(\hat{x})), P_1(x^*, F(x^*)), \\ &\quad \frac{1}{2}[P_1(x^*, F(x^*)) + d(\hat{x}, x^*)] \} \end{aligned}$$

which is a contradiction.

$$d(\hat{x}, P_1(x^*, F(x^*))) \leq d(\hat{x}, x^*) = r$$

This also contradicts. Hence the proof of theorem 3.2 is completed.

References

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