Fuzzy information in the interactive vector optimization methods

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The vector optimization problem can be written as follows:

$$\{f_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, m\} \longrightarrow \max$$

subject to

$$g_j(x_1, x_2, \ldots, x_n) \leqq C, \quad j = 1, 2, \ldots, p.$$

Here $f_i (i=1,2,\ldots,m)$ are the objective functions, g_j $(j=1,2,\ldots,p)$ the constraint functions.

When no additional information on the mutual relationships among the objectives is supplied, the problem is solved by finding all efficient solutions; a feasible solution $x=(x_1, x_2, \ldots, x_n)$ is called efficient if there exists no feasible solution y for which

$$f_i(x) \leqq f_i(y) \text{ for all } i,$$

$$f_{i_o}(x) < f_{i_o}(y) \text{ for at least one } i_o.$$

However from the practical point of view this concept of solution is rather unsuitable because the set of all efficient solutions is as a rule of great size and complex structure. To select such part of the set of all efficient solutions which can be considered as satisfactory solution we should have as much information as possible about the decision-maker's (DM's) preferences in the criterion space; in other words, we should know which combinations of objective functions values are preferred by the DM. One possible way of obtaining such information is to solve the problem by an interactive process.

The interactive process of solving a vector optimization problem has generally the form of an iterative cycle in which two phases are repeated:

a) computation phases performed by the analyst; their output is formed by a provisional solution given to the DM,

b) decision phases in which the DM forms his judgement about the

given provisional solution; if he is not satisfied he gives to the analyst some information about his preferences on the basis of which the analyst can compute a new solution more suitable to the DM.

The provisional solutions given to the DM in the individual steps of the procedure represent a certain compromise between individual objectives; therefore, they are often called compromise solutions. Most frequently they are computed either by maximizing the weighted sum of objective functions or by minimizing the distance from an ideal point, i.e. from a point in criterion space whose components are maxima of individual objective functions considered separately.

Let us turn our attention now to the information required by the analyst from the DM as a result of the decision phases of the process. Almost all interactive procedures ask in the first place whether the last compromise solution obtained is satisfactory to the DM; if it is not, other information on the preference structure at least in the neighbourhood of the known solution is required.

In some interactive procedures it is assumed that the structure of DM's preferences can be described by indifference hypersurfaces like in the theory of utility. The information required from the DM should in this case help to identify these hypersurfaces at least "near" to the compromise solution. The DM must give (or select from the list of) the rates of substitution between the objective functions. This is the case of e.g. Geoffrion or Zionts-Wallenius method.

In the second group of interactive procedures it is assumed that the DM can assess the values of individual objective functions reached by the compromise solution and state which of them he finds satisfactory. Further information of feasible or desired limits on objective functions values are then required. The majority of known interactive procedures belongs to this group; the most famous is the STEM method and its various modifications.

The third group consists of interactive procedures which do not use any explicit assumption about DM's preferences. When such a procedure applies, the analyst gives to the DM the list of several provisional solutions and the DM chooses the one which

suits him best. On the basis of his choice, further set of pro-
visional solutions is computed, from which the DM again choo-
ses and so on. The most famous procedure of this group is the
one suggested by Steuer.

Interactive processes of vector optimization therefore re-
quire in most cases from the DM quantitative information on his
preferences in the criterion space. Supplying such information
can be a very difficult task, especially for the following rea-
sons:

- the DM's preference structure need not be unambiguously given,
- the DM need not be one individual; the decision may be made
  by a collective body whose members can possess different pre-
  ference structures.

A means to express a vague structure of DM's preferences pre-
sents itself in the apparatus of fuzzy sets. If the DM is not
willing or able to give a particular numerical value of (say)
rates of substitution which is required from him he can give a
fuzzy set of such values.

In the case when the fuzzy information from the DM to the
analyst concerns the rates of substitution, the corresponding
provisional solution can be found by means of parametric program-
ming. This approach can be applied above all in the case when
all constraints and objective functions are linear; otherwise,
especially for non-convex problems, the corresponding paramet-
ric problems can be solved only with a great difficulty.

In the case when the fuzzy information from the DM concerns
the satisfactory values of individual objective functions the
problem is in some sense simpler. For some or all objective fun-
ctions f_k now the DM gives fuzzy sets V_k of their satisfactory
values; the fuzzy set V of satisfactory vectors in criterion space
can then be derived as their Cartesian product. If the sets V_k
are given for all $k = 1,2,\ldots,m$, we get directly the fuzzy set
of provisional solutions; otherwise it is necessary to optimize
on V a suitably chosen aggregating function, e.g. to minimize the
function

$$\hat{z}(x) = \max_k \left| f_k(x) - z_k^* \right|$$

where z_k^* denotes the maximal (ideal) value of the k-th objective function. This problem can be solved by the fuzzy optimization approach.

In all cases, therefore, the fuzzy information supplied by the DM leads to the construction of a fuzzy set of provisional solutions (instead of one solution which we have in the non-fuzzy case). The problem now arises what information about this fuzzy set should be sent back to the DM. We have essentially the three following possibilities:

a) The DM is given only one provisional solution, namely that with the maximal value of the membership function.

b) The DM gets all provisional solutions obtained or at least the sample of them which is sufficiently representative, together with their corresponding values of the membership function.

c) The DM is given all the provisional solutions for which the values of the membership function are bigger than a given limit, or the representative sample of such solutions.

Of the three possibilities mentioned, the alternative a) is the simplest one; however, a rather big part of the information previously given by the DM is not utilized. The alternative b) on the other hand loses no information about the fuzziness of DM's preferences but the volume of information the DM is given in each iteration of the process may well be too large for him. Alternative c) often represents a suitable compromise between the possibilities a) and b).