Vaguely Formulated Coefficients of Linear Inequalities

Jaroslav Ramík, Josef Římánek

Systems of linear inequalities with coefficients that are not given fixed are examined. Two cases are considered: systems whose coefficients gain their values from a given set with the same possibility of realization of each value and systems with fuzzy coefficients. Various types of solutions of such systems are defined and characterized.

1. Introduction

Solving problems of mathematical economy we often encounter the problem of finding nonnegative solutions of a system of inequalities

$$A\,x \leq b\ .$$

The matrix A and the vector b are usually supposed to be fixed characteristics of the modelled reality. Nevertheless, description of many real situations leads to systems with coefficients that are not given uniquely but are variable in a certain sense.

Such systems will be considered in this paper. Firstly, we mention the situation when coefficients gain values from a given set and possibilities of realization are equal for all values. We introduce two types of solutions and show some possibilities how to characterize sets of these solutions. Thereafter we consider systems whose coefficients are given as fuzzy matrices. For this case, that in fact is a generalization of the previous one, we again define two types of solutions and mention some of their properties.

This paper represents just a brief account of some results obtained by the authors. Therefore, proofs are ommited and motivation is only sketched.

2. Systems with variable coefficients

2.1. Formulation of the problem

The following system of inequalities will be examined:

$$(1) \qquad \begin{aligned} A\,x &\leq b\,, \\ x &\geq 0\,. \end{aligned}$$

with A being an m by n matrix, x being an n-vector, b being an m-vector. The matrix A is not fixed, we only know that it is an element of a given set $\mathcal{A} \subset E_{mn}$ /the set of all m by n matrices is considered to be identical with the Euclidean space E_{mn}/. Without loss of generality the vector b is considered fixed /the case of b variable can be carried over to the above mentioned case/. We consider two types of solutions of the system (1) and define the following two sets:

$$X_{\forall} = \{ x \in E_n;\ x \geq 0\,,\ \forall A \in \mathcal{A}\ [A\,x \leq b] \}\,,$$

$$X_{\exists} = \{ x \in E_n;\ x \geq 0\,,\ \exists A \in \mathcal{A}\ [A\,x \leq b] \}\,,$$

calling a vector $x \in X_{\forall}$ <u>the pesimistic solution of the system (1)</u> and a vector $x \in X_{\exists}$ <u>the optimistic solution of the system (1)</u>.

<u>Example.</u> The system (1) is often studied for a matrix interval $\mathcal{A}$:

$$\mathcal{A} = \{ A \in E_{mn}\ ;\quad \underline{A} \leq A \leq \overline{A} \}\,,$$

with $\underline{A}$, $\overline{A}$ being given matrices / " $\leq$ " is the "naturally" defined partial ordering of E_{mn}/. Another case is a finite set

$$\mathcal{A} = \{ A_1,\ A_2,\ \ldots\ ,A_s \}\ .$$

2.2. Description of solutions

As it can be easily seen, $X_{\forall}$ is a convex set while $X_{\exists}$ is generally not the case.

Let us define

$$\overline{X}_{\forall} = \{ x \in E_n\ ;\quad x \geq 0\ ,\ \ \forall A \in \overline{co}\ \mathcal{A}[A\ x \leq b] \}\,,$$

$$\overline{X}_{\exists} = \{ x \in E_n\ ;\quad x \geq 0\ ,\ \ \exists A \in \overline{co}\ \mathcal{A}[A\ x \leq b] \}\,,$$

where $\overline{co}\,\mathcal{A}$ means the closed convex closure of the set $\mathcal{A}$ in E_{mn}.

Then following assertion holds.

Assertion 1. Let $\mathcal{A} \subset E_{mn}$. Then

(2) $\quad X_{\forall} = \overline{X}_{\forall}$,

(3) $\quad X_{\exists} \subset \overline{X}_{\exists}$.

Generally, the inclusion opposite to that of relation (3) is not valid. In order to obtain a result similar to that of relation (2) , the following concept is introduced.

We say that a set $\mathcal{A} \subset E_{mn}$ has independent rows if there are sets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$, $\mathcal{A}_i \subset E_n$ for $i=1,2,\ldots,m$ such that $\mathcal{A}$ is the Cartesian product of $\mathcal{A}_i$:

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \ldots \times \mathcal{A}_m .$$

Assertion 2. Let $\mathcal{A} \subset E_{mn}$, $\mathcal{A}$ has independent rows. Then

(4) $\quad X_{\exists} = \overline{X}_{\exists}$.

Moreover, if $\mathcal{A}_i$, $i=1,2,\ldots,m$ are convex, then $X_{\exists}$ is also convex.

Remark. According to (2) and (4) we may confine ourselves to closed convex sets $\mathcal{A}$. The closed convex closure of $\mathcal{A}$ may be of far more simple structure than the original set $\mathcal{A}$. If e.g. $\overline{\text{co}}\,\mathcal{A}$ is a convex polyhedron than /as it can be easily verified/ $X_{\forall}$, $X_{\exists}$ are fully determined by their vertices.

3. Systems with fuzzy coefficients.

We shall deal with the system

(5)
$$\begin{aligned} \mathcal{A}x &\leq b , \\ x &\geq 0 \end{aligned}$$

with $\mathcal{A}$ being an m by n fuzzy matrix with the membership function

$$\mu_{\mathcal{A}} : E_{mn} \longrightarrow \langle 0 , 1 \rangle .$$

Denote the β - level-set of $\mathcal{A}$ / $\beta \in \langle 0,1 \rangle$ / by $\mathcal{A}_\beta$:

$$\mathcal{A}_\beta = \{ A \in E_{mn} ; \quad \mu_{\mathcal{A}}(A) \geq \beta \}$$

and set

$$X_{\forall}^{\beta} = \{ x \in E_n ; x \geq 0 , \forall A \in \mathcal{A}_{\beta} [A x \leq b] \} ,$$
$$X_{\exists}^{\beta} = \{ x \in E_n ; x \geq 0 , \exists A \in \mathcal{A}_{\beta} [A x \leq b] \} .$$

The fuzzy set $\mathcal{X}_{\forall}$ in E_n given by the membership function

$$(6) \quad \varphi_{\forall} : E_n \longrightarrow \langle 0 , 1 \rangle , \quad \varphi_{\forall}(x) = \lambda(\{ \beta \in \langle 0 , 1 \rangle ; x \in X_{\forall}^{\beta} \})$$

/where $\lambda(M)$ means the Lebesque measure of the set $M \subset E_1$ / will be called the pesimistic fuzzy solution of the system (5) .
The fuzzy set $\mathcal{X}_{\exists}$ in E_n with the membership function

$$(7) \quad \varphi_{\exists} : E_n \longrightarrow \langle 0 , 1 \rangle , \quad \varphi_{\exists}(x) = \lambda(\{ \beta \in \langle 0 , 1 \rangle ; x \in X_{\exists}^{\beta} \})$$

will be called the optimistic fuzzy solution of the system (5) .

Remark. The problem (5) is a generalization of the problem (1) as it can be seen by setting

$$\mu_{\mathcal{A}}(A) = \begin{cases} 0 & \text{for } A \notin \mathcal{A} \\ 1 & \text{for } A \in \mathcal{A} . \end{cases}$$

Remark. Immediately from (6) and (7) we have

$$\varphi_{\forall}(x) = 1 - \inf \{ \beta \in \langle 0 , 1 \rangle ; x \in X_{\forall}^{\beta} \} ,$$
$$\varphi_{\exists}(x) = \sup \{ \beta \in \langle 0 , 1 \rangle ; x \in X_{\exists}^{\beta} \} .$$

Remark. Consider α-level-sets of $\mathcal{X}_{\forall}$, $\mathcal{X}_{\exists}$ / $\alpha \in \langle 0 , 1 \rangle$ / .
It holds:

$$(8) \qquad \mathcal{X}_{\forall}^{\alpha} = X_{\forall}^{1-\alpha} , \qquad \mathcal{X}_{\exists}^{\alpha} = X_{\exists}^{\alpha} .$$

Conclusion.

(i) $\mathcal{X}_{\forall}$ is a convex fuzzy set, since all its level sets are convex.

(ii) Let the fuzzy matrix $\mathcal{A}$ be given by the membership function

$$\mu_{\mathcal{A}}(A) = \min_i \varphi_i(a_i) ,$$

with $A = \{a_1, a_2, \dots, a_n\}$, $a_i \in E_m$ /a_i are the columns of A / and

$$\varphi_i : E_m \longrightarrow \langle 0, 1\rangle$$

being quasi concave membership functions. Then $\mathcal{X}_3$ is a convex fuzzy set as it follows from (8) and from the Assertion 2.

Authors address:

Jaroslav Ramík

Iron and Steel Institute
Hasičská 52
705 00 Ostrava 3

Czechoslovakia

Josef Římánek

Mining and Metallurgical University
Osvoboditelů 33
701 00 Ostrava 1

Czechoslovakia