A REDUCTION OF FUZZY AUTOMATA

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Among various types of automata, as is well-known, are deterministic, nondeterministic and probabilitistic automata. W.G.Wee [2] proposed one another type of automata which he named fuzzy automaton. Fuzzy automata include basic types of known automata. A fuzzy automaton is a type of automaton which will transite from a state to another state or the same state via the branch whose membership function is the largest one among those of all branches diverging from the state when an input is applied.

We consider the following (Moore) type of fuzzy automaton. A <u>fuzzy automaton</u> is the following couple,

$$\mathfrak{M} = (S, \Sigma, \pi, \{F(\sigma) : \sigma \in \Sigma\}, G),$$

where

(1) S is a finite set of states,

(2) Σ is a finite set of inputs (<u>input alphabet</u>) ,

(3) π is an <u>initial state designator</u> , i.e. $\pi \underset{\sim}{\subseteq} S$ (the symbol $\underset{\sim}{\subseteq}$ denotes a fuzzy set in S)

(4) for each $\sigma \in \Sigma$, $F(\sigma)$ is a fuzzy matrix of order n (the <u>fuzzy transition matrix</u>) ,i.e. $F(\sigma) \underset{\sim}{\subseteq} S \times S$,

(5) G is a subset in S, the <u>set of final states</u>.

Every fuzzy set A in S we call <u>fuzzy state</u> of $\mathfrak{M}$. If an input signal $\sigma \in \Sigma$ is accepted by $\mathfrak{M}$, the present fuzzy state A(old) will change on the fuzzy state $A(\text{new}) = A \circ F(\sigma)$, where "o" is an composition rule of fuzzy relations.

In this note we deal with so called <u>optimistic fuzzy automaton</u> only, i.e. "o" is a min-max product.

The principal identification of fuzzy automaton may be done by its output function $f_{\mathfrak{A}}$,

$$f_{\mathfrak{A}} : \Sigma^* \longrightarrow [0,1]$$

$$f_{\mathfrak{A}}(x) = \Pi \circ F(x) \circ G \ , \ x \in \Sigma^*$$

where for $x = \sigma_1 \ldots \sigma_m \in \Sigma^*$ we have $F(x) = F(\sigma_1) \circ \ldots \circ F(\sigma_m)$, and G is considered as a trivial fuzzy set , $G \subsetneq S$. Clearly, $f_{\mathfrak{A}}(x)$ is designated as the grade of transition of $\mathfrak{A}$, when started with initial distribution Π over S to enter a state in G after scanning the input sequence x. Then an input sequence x is said to be accepted by $\mathfrak{A}$ with grade $f_{\mathfrak{A}}(x)$.

In this note we deal with a reduction of a family of states of $\mathfrak{A}$ in such a way that for a newly created automaton $\overline{\mathfrak{A}}$ with a reduced set of states its output function will be approximately the same as the original one. For classical automata this problem was generally solved in a procedure of minimalization that uses a notion of <u>homomorphism</u> of automata. We show, at first, that this procedure is not effective in a family of fuzzy automata.

It should be observed that in a family of fuzzy automata it is possible to introduce the notion of a homomorphism that is closely related to the original one.

<u>DEFINITION 1.</u> Let $\mathfrak{A}_i = (S_i, \Sigma_i, \Pi_i, \{F^i(\sigma) : \sigma \in \Sigma_i\}, G_i)$, $i=1,2$, be fuzzy automata. A pair (α, β) is called a

homomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$, $(\alpha,\beta):\mathfrak{A}_1 \longrightarrow \mathfrak{A}_2$, if $\alpha : S_1 \longrightarrow S_2$, $\beta : \Sigma_1 \longrightarrow \Sigma_2$ are maps such that the following diagrams are commutative.

$$\begin{tikzcd} S_1 \arrow[rr, "\alpha"] \arrow[dr, "\pi_1"'] & & S_2 \arrow[dl, "\pi_2"] \\ & {[0,1]} & \end{tikzcd} \qquad \begin{tikzcd} S_1 \arrow[rr, "\alpha"] \arrow[dr, "G_1"'] & & S_2 \arrow[dl, "G_2"] \\ & {[0,1]} & \end{tikzcd}$$

$$\begin{tikzcd} S_1 \times S_1 \times \Sigma_1 \arrow[rr, "\alpha \times \alpha \times \beta"] \arrow[dr, "F_1"'] & & S_2 \times S_2 \times \Sigma_2 \arrow[dl, "F_2"] \\ & {[0,1]} & \end{tikzcd}$$

The basic relation between homomorphisms and output functions expresses the following proposition.

PROPOSITION 1. Let $(\alpha,\beta) : \mathfrak{A}_1 \longrightarrow \mathfrak{A}_2$ be a homomorphism such that α is a surjection. Then the following diagram commutates, where β^* is the canonical extension of β .

$$\begin{tikzcd} \Sigma_1^* \arrow[rr] \arrow[dr, "f_{\mathfrak{A}_1}"'] & & \Sigma_2^* \arrow[dl, "f_{\mathfrak{A}_2}"] \\ & {[0,1]} & \end{tikzcd}$$

P r o o f. Since (α,β) is a homomorphism, we have $F^1_{s,t}(\sigma) = F^2_{\alpha s,\alpha t}(\beta\sigma)$ for every $s,t \in S_1$, $\sigma \in \Sigma_1$. We show that for every $x \in \Sigma_1^*$ the relation $F^1_{s,t}(x) = F^2_{\alpha s,\alpha t}(\beta^* x)$ holds, where F^i we identify with the canonical extension $F^i : S_i \times S_i \times \Sigma_i^* \longrightarrow [0,1]$ of F^i . The proof of this will be done by induction with respect to a length $|x|$ of a word x.

The proposition holds for $x \in \Sigma_1^*$ such that $|x| = 1$, i.e. $x \in \Sigma_1$. Let $y \in \Sigma_1^*$ be such that $y = x.\sigma$, $|x| \leq n$, $\sigma \in \Sigma_1$. Then

$$F^1_{s,t}(x,\sigma) = \bigvee_{p \in S_1} (F^1_{s,p}(x) \wedge F^1_{p,t}(\sigma)) =$$

$$= \bigvee_{p \in S_1} (F^2_{\alpha s,\alpha p}(\beta^* x) \wedge F^2_{\alpha p,\alpha t}(\beta\sigma)) = \bigvee_{q \in S_2} (F^2_{\alpha s,q}(\beta^* x) \wedge$$

$$\wedge F^2_{q,\alpha t}(\beta\sigma)) = F^2_{\alpha s,\alpha t}(\beta^* x.\beta\sigma) = F^2_{\alpha s,\alpha t}(\beta^* y) .$$

Now, let for every $x \in \Sigma_1^*$, $\tilde{x}$ is a fuzzy state such that $\tilde{x} = \pi_1 \circ F^1(x)$. Then for every $x \in \Sigma_1$ the following diagrame commutates.

$$\begin{tikzcd} S_1 \arrow[rr, "\alpha"] \arrow[dr, "\tilde{x}"'] & & S_2 \arrow[dl, "\widetilde{\beta^* x}"] \\ & {[0,1]} & \end{tikzcd}$$

In fact, using the above relation we get

$$\tilde{x}(s) = (\pi_1 \circ F^1(x))(s) = \bigvee_{p \in S_1} (\pi_1(p) \wedge F^1_{p,s}(x)) =$$

$$= \bigvee_{p \in S_1} (\pi_2(\alpha p) \wedge F^2_{\alpha p,\alpha s}(\beta^* x)) = \bigvee_{t \in S_2} (\pi_2(t) \wedge F^2_{t,\alpha s}(\beta^* x)) =$$

$$(\pi_2 \circ F^2(\beta^* x))(\alpha s) = \widetilde{\beta^* x}\,(\alpha s).$$

Finally, for $x \in \Sigma_1^*$ we have (here $f_i := f_{\mathfrak{M}_i}$)

$$f_1(x) = \pi_1 \circ F^1(x) \circ G_1 = \bigvee_{s \in S_1} ((\pi_1 \circ F^1(x))(s) \wedge G_1(s)) =$$

$$= \bigvee_{s \in S_1} (\tilde{x}(s) \wedge G_1(s)) = \bigvee_{s \in S_1} (\widetilde{\beta^* x}(\alpha s) \wedge G_2(\alpha s)) =$$

$$\bigvee_{t \in S_2} (\widetilde{\beta^* x}(t) \wedge G_2(t)) = \pi_2 \circ F^2(\widetilde{\beta^* x}) \circ G_2 = f_2(\beta^* x). \qquad \blacksquare$$

Now, an analogy of classical construction of minimalization of fuzzy automaton requires to construct for any fuzzy automaton $\mathfrak{M} = (S, \Sigma, \pi, \{F(\sigma)\}, G)$ equivalence relations R, T on the sets S, Σ, respectively, such that the canonical maps $\alpha : S \longrightarrow S_1 = S/R$, $\beta : \Sigma \longrightarrow \Sigma_1 = \Sigma/T$ create homomorphism $(\alpha, \beta) : \mathfrak{M} \longrightarrow \overline{\mathfrak{M}}$, where $\overline{\mathfrak{M}}$ is a fuzzy automaton with the input alphabet Σ_1 and states S_1. The natural question arising here is the following : What are the necessary properties of equivalence relations R and T ? Using the definition of homomorphism it is easy to see that the following properties must be satisfied.

(1) If $(s_1, s_2) \in R$ then $\pi(s_1) = \pi(s_2)$, $G(s_1) = G(s_2)$.

(2) If (s_1, s_2), $(t_1, t_2) \in R$, $(\sigma_1, \sigma_2) \in T$ then

$$F_{s_i, t_j}(\sigma_1) = F_{s_u, t_v}(\sigma_2), \quad i,j,u,v \in \{1,2\}.$$

These conditions are rather strong and in almost all real fuzzy automata it is not possible to construct equivalence relations which satisfy these conditions.

For these reasons we abandon the exactness in the definition of reduction of fuzzy automata and return to approximation. With this modification the reduction of a fuzzy automaton may be done as follows.

Let $\varepsilon > 0$ be a real number (the <u>level of identification</u>)

and let $R\ (= R_{\varepsilon})$ be an equivalence relation on S with the following additional properties (here $\bar{s}$ denotes the element of the factor set S/R such that $s \in \bar{s}$) :

For every $s \in S$, $t, r \in \bar{s}$ the following hold :

(1) For every $p \in S$, $\sigma \in \Sigma$ we have $|F_{r,p}(\sigma) - F_{t,p}(\sigma)| < \varepsilon$,
$|F_{p,r}(\sigma) - F_{p,t}(\sigma)| < \varepsilon$,

(2) $|\pi(r) - \pi(t)| < \varepsilon$

(3) $|G(r) - G(t)| < \varepsilon$ (i.e. $G(r) = G(t)$).

In real situations a decomposition of S with these properties (for various ε) may be , clearly, done. Moreover, we construct a new fuzzy automaton $\bar{\mathfrak{U}}_{\varepsilon}$ as follows. We set

$$\bar{S}_{\varepsilon} = S/R_{\varepsilon} ,$$

$$\bar{\pi}(\bar{s}) = (1/|\bar{s}|) \sum_{t \in \bar{s}} \pi(t)$$, where $|\bar{x}|$ is the cardinality of $\bar{x} \subseteq S$,

$$\bar{G}(\bar{s}) = \begin{cases} 1, \text{ iff } \bar{s} \cap G \neq \emptyset \\ 0, \text{ iff } \bar{s} \cap G = \emptyset , \end{cases}$$

$$\bar{F}_{\bar{s},\bar{t}}(\sigma) = (1/|\bar{s}| \cdot |\bar{t}|) \sum_{i \in \bar{s}, j \in \bar{t}} F_{i,j}(\sigma) .$$

Let $\bar{\mathfrak{U}}_{\varepsilon} = (\bar{S}_{\varepsilon} , \Sigma , \bar{\pi} , \{\bar{F}(\sigma)\} , \bar{G})$. The main result of this note is the following theorem which expresses the relation between the output function f of $\mathfrak{U}$ and the output function $\bar{f}$ of $\bar{\mathfrak{U}}_{\varepsilon}$.

<u>THEOREM 1.</u> For every $x \in \Sigma^{*}$ we have

$$|f(x) - \bar{f}(x)| < \varepsilon(|x| + 2) .$$

In most real situations we deal with input words x with restricted length , $|x| \leqslant n_o$ .Then from a required assumption $|f(x) - \bar{f}(x)| < \delta$ we may compute a level of identification ε and construct a decomposition $\bar{S}_\varepsilon$ of S.

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